Published by Institute of Physics Publishing for SISSA/ISAS
Received: *May 9, 2003*
Revised: *September 29, 2003*
Accepted: *October 3, 2003*

Moduli stabilization from fluxes in a simple IIB orientifold

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Abstract: We study novel type-IIB compactifications on the T^6/Z_2 orientifold. This geometry arises in the T-dual description of type-I theory on T^6, and one normally introduces 16 space-filling D3-branes to cancel the RR tadpoles. Here, we cancel the RR tadpoles either partially or fully by turning on three-form flux in the compact geometry. The resulting (super)potential for moduli is calculable. We demonstrate that one can find many examples of $\mathcal{N} = 1$ supersymmetric vacua with greatly reduced numbers of moduli in this system. A few examples with $\mathcal{N} > 1$ supersymmetry or complete supersymmetry breaking are also discussed.

Keywords: Superstrings and Heterotic Strings, Superstring Vacua.

JHEP10(2003)007

 http://jhep.sissa.it/archive/papers/jhep102003007/jhep102003007.pdf

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1. Introduction

The study of Calabi-Yau orientifold compactifications of type-II string theory (or F-theory compactifications on Calabi-Yau fourfolds), with nontrivial background RR and NS fluxes through compact cycles of the Calabi-Yau manifold, is of interest for several reasons.

Conventional compactifications give rise to models which typically have many moduli. Understanding how these flat directions are lifted is important, both from the point of view of phenomenology and of cosmology. One expects the moduli to be lifted once supersymmetry is broken, but studying this in a calculable way in conventional compactifications has proved challenging so far. In contrast, compactifications with background RR and NS fluxes turned on give rise to a nontrivial low energy potential which freezes many of the Calabi-Yau moduli. Moreover, the potential is often calculable and as a result one can hope to study the stabilization of many moduli in a controlled manner in this setting. Flux-induced potentials for moduli have been discussed before in e.g. [1]–[7] (while a complementary "stringy" means of freezing moduli, by considering asymmetric orbifolds, has been discussed in, for example, [8]).

Compactifications with fluxes have also been proposed as a natural setting for warped solutions to the hierarchy problem [9], along the lines of the proposal of Randall and Sundrum [10]. The combination of fluxes and space filling D-branes which often need to be introduced for tadpole cancellation in these models leads to a nontrivial warped metric, with the scale of 4d Minkowski space varying over the compact dimensions. Examples of such models, with almost all moduli stabilized and exponentially large warping giving rise to a hierarchy, appeared in [5]. (See also [11]).

Finally, compactifications with fluxes also have interesting (and relatively unexplored) dual descriptions, via mirror symmetry and heterotic/type-II duality. Some examples of these dualities have been discussed in [4, 12].

In this paper, we explore in detail the simplest such compactification which admits supersymmetric vacua with nontrivial NS and RR fluxes: the compactification of type-IIB string theory on the T^6/Z_2 orientifold. The most familiar avatar of this model includes 16 D3-branes which cancel the RR charge of the 64 O3-planes at the 2^6 fixed points of the Z_2 action. However, one is free to replace some (or all) of the D3-branes with appropriate integral RR and NS 3-form fluxes $F_{(3)}$ and $H_{(3)}$. Given such a choice of integral fluxes, one can compute the low-energy superpotential governing the light fields. In a generic Calabi-Yau orientifold in IIB string theory, the periods which are required to determine W would only be computable as approximate expansions about various extreme points in moduli space, making any global and tractable expression for W difficult to obtain. A nice feature of the T^6/Z_2 case is that W is easily computable.

With the superpotential in control we can ask if there are $\mathcal{N} = 1$ supersymmetry preserving minima. It turns out that for generic choices of the fluxes supersymmetry is broken. By suitably choosing the fluxes, however, we find several examples which give rise to stable, $\mathcal{N} = 1$ supersymmetric ground states. In these minima, typically, the dilaton-axion, all complex structure moduli, and some of the Kähler moduli are stabilized. In addition, since some or all of the O3-plane charge is cancelled by the flux, fewer D3-branes are present, and the number of moduli coming from the open string sector is also reduced. The conventional IIB compactification on this T^6/Z_2 orientifold has 67 (complex) moduli.[1] Once fluxes are turned on, it is easy to find examples with far fewer moduli (~ 3 in the models we discuss here, and fewer in the class of models described in [5]).

The organization of this paper is as follows. In section 2, we review basic facts about vacua with flux and about the moduli of the T^6/Z_2 orientifold, and parametrize the possible choices of flux. In section 3, we discuss the constraints that must be imposed to find a supersymmetric vacuum, following [13, 14], and write down a formula for the superpotential as a function of the T^6 moduli. In section 4, we exhibit many examples which lead to $\mathcal{N} = 1$ supersymmetric solutions. We also analyze some cases which turn out to have $\mathcal{N} = 3$ supersymmetry and make some comments about finding the most general supersymmetric solution. In section 5 we discuss the conditions under which two apparently distinct solutions are nevertheless equivalent (using the reparametrization symmetries of the torus and U-duality). In section 6 we describe how, starting from a supersymmetric solution, additional physically distinct ones can be found using rescalings and $\mathrm{GL}(2,\mathbb{Z}) \times \mathrm{GL}(6,\mathbb{Z})$ transformations. In section 7, we derive the conditions which must be imposed on the $G_{(3)}$ flux to find $\mathcal{N} = 2$ supersymmetric solutions, and consider one illustrative example. section 8 contains some examples of nonsupersymmetric solutions. In section 9 we discuss the dynamics on the D3 branes which one should insert into many of our vacua, to saturate the D3 tadpole. We close with a brief description of directions for future research in section 10, and some important details are relegated to appendices A–D. Finally, for the JHEP version of this paper, we have appended a brief description of progress that has occurred since the preprint version of this paper was released. This appears as section 11 of the published paper.

While this work was in progress, we learned of a related work exploring novel 4d $\mathcal{N} = 3$ supersymmetric vacua which can be found from special flux configurations on T^6/Z_2 [15]. We are grateful to the authors of [15] for providing us with an early version of their paper, and for helpful comments.

[1] The model has 16 D3-branes each of which give rise to 3 moduli, in addition there are 19 moduli coming from the closed string sector.

2. Preliminaries

2.1 D3-brane charge from 3-form flux

The type-IIB supergravity action in Einstein frame is [16]

$$\begin{aligned} S_{\rm IIB} = & \frac{1}{2{\kappa_{10}}^2}\int d^{10}x\sqrt{-g}\left(R - \frac{\partial_M\phi\partial^M\phi}{2({\rm Im}\phi)^2} - \frac{G_{(3)}\cdot\bar{G}_{(3)}}{2\cdot 3!{\rm Im}\phi} - \frac{{\widetilde{F}_{(5)}}^2}{4\cdot 5!}\right) + \\ & + \frac{1}{2{\kappa_{10}}^2}\int\frac{C_{(4)}\wedge G_{(3)}\wedge\bar{G}_{(3)}}{4i{\rm Im}\phi} + S_{\rm local}\,. \end{aligned} \tag{2.1}$$

Here,

$$\phi = C_{(0)} + \frac{i}{g_s}\,,\qquad G_{(3)} = F_{(3)} - \phi H_{(3)}\,, \tag{2.2}$$

and

$$\widetilde{F}_{(5)} = F_{(5)} - \frac{1}{2}C_{(2)}\wedge H_{(3)} + \frac{1}{2}F_{(3)}\wedge B_{(2)}\,,\qquad \text{with}\quad *\widetilde{F}_{(5)} = \widetilde{F}_{(5)}\,, \tag{2.3}$$

where $F_{(3)} = dC_{(2)}$ and $H_{(3)} = dB_{(2)}$. If one compactifies on a six dimensional compact manifold, $\mathcal{M}_6$, and includes the possibility of space-filling D3-branes and O3-planes, then the equation of motion/Bianchi identity for the 5-form field strength is

$$d\widetilde{F}_{(5)} = d*\widetilde{F}_{(5)} = H_{(3)}\wedge F_{(3)} + 2{\kappa_{10}}^2\mu_3\rho_3^{\rm local}\,. \tag{2.4}$$

Here μ_3 is the charge density of a D3-brane and $\rho_3^{\rm local}$ is the number density of local sources of D3-brane charge on the compact manifold. We can integrate this equation over $\mathcal{M}_6$ to give the condition

$$\frac{1}{2{\kappa_{10}}^2\mu_3}\int_{\mathcal{M}_6} H_{(3)}\wedge F_{(3)} + Q_3^{\rm local} = 0\,. \tag{2.5}$$

In condition (2.5), $Q_3^{\rm local}$ is the sum of contibutions $+1$ for each D3-brane and $-1/4$ for each normal O3-plane. As discussed in [17] and [18], there are actually three other types of O3-plane, each characterized by the presence of discrete RR and/or NS flux at the orientifold plane. These exotic O3-planes each contribute $+1/4$ to $Q_3^{\rm local}$.

We will be interested in the case that $\mathcal{M}_6$ is the T^6/Z_2 orientifold. There are 2^6 O3-planes in this compactification, with a total contribution of $-16+\frac{1}{2}N_{{\rm O3}'}$ units of D3-brane charge to $Q_3^{\rm local}$, where $N_{{\rm O3}'}$ is the number of exotic O3-planes. Therefore, (2.5) takes the form

$$\frac{1}{2}N_{\rm flux} + N_{\rm D3} + \frac{1}{2}N_{{\rm O3}'} = 16\,. \tag{2.6a}$$

Here

$$N_{\rm flux} = \frac{1}{(2\pi)^4(\alpha')^2}\int_{T^6} H_{(3)}\wedge F_{(3)}\,. \tag{2.6b}$$

The factor of 1/2 multiplying $N_{\rm flux}$ compensates for the fact that the integration is over T^6 rather than T^6/Z_2. We have also replaced the prefactor, $1/(2{\kappa_{10}}^2\mu_3)$, with its explicit value in terms of α'. It is clear from (2.6) that appropriately chosen three-form fluxes can carry D3-brane charge. The fluxes obey a quantization condition

$$\frac{1}{(2\pi)^2\alpha'}\int_\gamma F_{(3)} = m_\gamma\in\mathbb{Z}\,,\qquad \frac{1}{(2\pi)^2\alpha'}\int_\gamma H_{(3)} = n_\gamma\in\mathbb{Z}\,, \tag{2.7}$$

where γ is an arbitrary class in $H_3(T^6,\mathbb{Z})$. There is a subtlety in arguing that these are the correct quantization conditions for T^6/Z_2 [15].[2] This is because there are additional three cycles in T^6/Z_2, which are not present in the covering space T^6. If some of the integers m_γ (n_γ) are odd, additional discrete RR (NS) flux needs to be turned on at appropriately chosen orientifold planes to meet the quantization condition on these additional cycles. (See appendix A for more discussion of this condition). In practice it is quite non-trivial to turn on the required discrete flux in a consistent manner without violating the charge conservation condition (2.6). We will avoid these complications in this paper, by restricting ourselves to cases where m_γ, n_γ are even integers, and by not including any discrete flux at the orientifold planes.

Finally, $G_{(3)}$ obeys an imaginary self-duality (ISD) condition, $*_6 G_{(3)} = iG_{(3)}$, as will be shown in the next section. This condition implies that the 3-form flux contributes positively to the total D3-brane charge. To see this note that the ISD condition implies that

$$\frac{*_6 H_{(3)}}{g_s} = -(F_{(3)} - C_{(0)} H_{(3)})\,. \tag{2.8}$$

Since $H_{(3)} \wedge F_{(3)} = H_{(3)} \wedge (F_{(3)} - C_{(0)} H_{(3)})$, we learn that[3]

$$\begin{aligned} \int_{\mathcal{M}_6} H_{(3)} \wedge F_{(3)} &= -\frac{1}{g_s}\int_{\mathcal{M}_6} H_{(3)} \wedge *_6 H_{(3)} \\ &= \frac{1}{g_s}\frac{1}{3!}\int_{\mathcal{M}_6} \sqrt{g_{\mathcal{M}_6}}\, {H_{(3)}}^2 > 0\,. \end{aligned} \tag{2.9}$$

Therefore, in the presence of nontrivial RR and NS fluxes which carry nonzero $N_{\rm flux}$, the number of D3 branes required to saturate (2.6) will always be fewer than 16.[4] In fact, in some models, one can entirely cancel the tadpole with fluxes.

2.2 The scalar potential from 3-form flux

Turning on three-form fluxes gives rise to a potential for some of the moduli. The four dimensional effective theory has a term of the form [5]

$$\mathcal{L}_G = \frac{1}{4{\kappa_{10}}^2}\int_{\mathcal{M}_6} d^6y \frac{G_{(3)} \wedge *_6 \bar{G}_{(3)}}{{\rm Im}\phi}\,, \tag{2.10}$$

which arises from the $G_{(3)} \cdot \bar{G}_{(3)}$ term in the ten dimensional action (2.1). To understand why this term gives rise to a potential for some moduli it is useful to write

$$G_{(3)} = G^{\rm ISD} + G^{\rm IASD}\,, \tag{2.11}$$

where

$$\begin{aligned} *_6 G^{\rm ISD} &= +iG^{\rm ISD}\,, \\ *_6 G^{\rm IASD} &= -iG^{\rm IASD}\,. \end{aligned} \tag{2.12}$$

[2]We are indebted to A. Frey and J. Polchinski for pointing out this subtlety.

[3]In the conventions of [5], $H_{(3)} \wedge *_6 H_{(3)} = -\frac{1}{3!} H_{mnp}H^{mnp}$ Vol, where m,n,p are real coordinates on $\mathcal{M}_6$ and Vol is the volume form.

[4]We do not allow the presence of anti D3-branes, since our main interest is SUSY solutions. Some aspects of non-supersymmetric vacua with anti D3-branes and fluxes have recently been described in [19].

Then,

$$\mathcal{L}_G = \frac{1}{2\kappa_{10}{}^2\mathrm{Im}\phi}\int_{\mathcal{M}_6} G^{\mathrm{IASD}} \wedge *_6\bar{G}^{\mathrm{IASD}} - \frac{i}{4\kappa_{10}{}^2\mathrm{Im}\phi}$$

$$\int_{\mathcal{M}_6} G_{(3)} \wedge \bar{G}_{(3)} = \mathcal{V}_{\mathrm{scalar}} + \text{topological}\,. \tag{2.13}$$

The second term in (2.13) is topological. It is proportional to N_{flux} (2.6) and independent of moduli. One expects on general grounds that three-form flux configurations, which give rise to D3-brane charge, should also lead to D3-brane tension. This contribution to D3-brane tension is accounted for by the second term.

The first term in (2.13) gives rise to the scalar potential and is central to this paper. It is positive semidefinite and vanishes when the flux meets the imaginary self-duality condition. The moduli dependence enters in two ways. First, $G_{(3)}$ depends on the axion-dilaton (2.2). Second, the decomposition of $G_{(3)}$ into ISD and IASD parts, depends on some metric moduli. Requiring that $G_{(3)}$ is imaginary self dual fixes many of these moduli.

2.3 IIB on the orientifold T^6/Z_2

Let us now focus on IIB string theory compactified on a T^6/Z_2 orientifold. The six transverse directions will be denoted as x^i, y^i, $i = 1,\ldots,3$. The orientifold action can be denoted as $\Omega R(-1)^{F_L}$, where R stands for a reflection of all of the compactified dimensions $(x^i, y^i) \to -(x^i, y^i), i = 1,\ldots,3$. In fact the model is related to the type-I theory compactified on T^6 by six T-dualities along all the compactified directions. It preserves $\mathcal{N} = 4$ supersymmetry, i.e., 16 supercharges.

The massless fields after compactification arise from the massless fields in the IIB ten dimensional supergravity theory. The bosonic fields in the ten dimensional theory are the metric g_{MN}, the NS 2-form $B_{(2)}$, the RR fields $C_{(2)}$,$C_{(4)}$, and the dilaton-axion $\phi, C_{(0)}$. Their transformation properties under $\Omega(-1)^{F_L}$ are as follows:

$$\begin{array}{rcc} & \Omega & (-1)^{F_L} \\ g_{MN} & + & + \\ B_{(2)} & - & + \\ C_{(2)} & + & - \\ C_{(4)} & - & - \\ C_{(0)} & - & - \\ \phi & + & + \end{array} \tag{2.14}$$

The resulting massless bosonic fields are then:

$$\begin{array}{rl} g_{\mu\nu} & 1 \text{ graviton} \\ g_{ab} & 21 \text{ scalars} \\ (B_{(2)})_{a\mu} & 6 \text{ gauge bosons} \\ (C_{(2)})_{a\mu} & 6 \text{ gauge bosons} \\ (C_{(4)})_{abcd} & 15 \text{ scalars} \\ C_{(0)} & 1 \text{ scalar} \\ \phi & 1 \text{ scalar} \end{array} \tag{2.15}$$

We see that the massless fields which survive the orientifold projection are the graviton, 12 gauge bosons and 38 scalars, plus their fermionic partners. These are organized into representations of $\mathcal{N} = 4$ supergravity as follows. The graviton, six gauge bosons and the axion-dilaton along with their fermionic partners, lie in a supergravity multiplet [16]. In addition there are six vector multiplets each containing a gauge boson, six scalars and their fermionic partners. Thus, in the absence of 3-form flux, the moduli space of T^6/Z_2 compactifications is parametrized by 38 scalars. When 3-form flux is turned on, some of the scalars from $C_{(4)}$ become charged, which means that they obtain Stuckelberg type kinetic terms $\sim (\partial_\mu \lambda + m A_\mu)^2$, where m is determined by the flux. For generic $\mathcal{N} = 1$ solutions, one can show that twelve of these scalars are eaten by gauge fields though the Higgs mechanism. (see, for example, [20] or [15] for related discussions in somewhat different contexts). Six of these twelve scalars are partners of metric Kähler moduli which also get heavy. The remaining three scalars from $C_{(4)}$ pair up with three metric Kähler moduli to form three $\mathcal{N} = 1$ chiral multiplets which survive in the low-energy theory.

In the T-dual of type-I theory on T^6, one would also include 16 D3-branes, each with a worldvolume $\mathcal{N} = 4$ vector multiplet. We will ignore any brane worldvolume fields for now, and briefly discuss the physics on the branes we must introduce in section 9.

We discussed above that turning on fluxes leads to a potential on moduli space. It is important to note that although some of the moduli will gain a mass from this potential, the effective field theory keeping only the fields (2.15) from the closed string sector (plus any massless open string fields, if branes are introduced) is valid. This is because the masses generated by the flux-induced potential will scale like $m \sim \alpha'/R^3$, where we have assumed an isotropic torus of size $\sim R$. The KK modes on the Calabi-Yau geometry have masses that scale like $m_{KK} \sim 1/R$, so if we work at sufficiently large radius (where our supergravity considerations are most valid in any case), $m \ll m_{KK}$, and we are justified in truncating to the field theory of the modes (2.15).

It is helpful to regard the torus as a complex manifold and organize the various moduli accordingly. Nine of the twenty-one scalars that arise from the ten-dimensional metric correspond to Kähler deformations, while the remaining twelve scalars correspond to complex structure deformations.

An essential difference between the six-torus and a Calabi-Yau three-fold is the following. For a generic CY_3, Yau's theorem implies that any complex structure or Kähler deformation corresponds to a nontrivial deformation of the Ricci-flat metric. This is not true for the six-torus or the T^6/Z_2 case at hand. In this case, as we will see below, the complex structure is specified by nine complex parameters. Three of these parameters correspond to deformations of the complex structure at fixed metric.

2.4 The complex structure of a torus

Nine complex coordinates are needed to describe the complex structure of T^6. We will use the explicit parametrization discussed in [21], which is summarized below. Let x^i, y^i, $i = 1, \ldots, 3$ be six real coordinates on T^6 which are periodic, $x^i \equiv x^i + 1$, $y^i \equiv y^i + 1$, and take the holomorphic 1-forms to be $dz^i = dx^i + \tau^{ij} dy^j$. The complex structure is completely specified by the period matrix τ^{ij}. We choose the orientation[5]

$$\int dx^1 \wedge dx^2 \wedge dx^3 \wedge dy^1 \wedge dy^2 \wedge dy^3 = 1\,, \tag{2.16}$$

and use the following basis of $H^3(T^6, \mathbb{Z})$:

$$\begin{aligned}
\alpha_0 &= dx^1 \wedge dx^2 \wedge dx^3\,, \\
\alpha_{ij} &= \frac{1}{2}\epsilon_{ilm} dx^l \wedge dx^m \wedge dy^j\,, \qquad\qquad 1 \le i,j \le 3\,, \\
\beta^{ij} &= 7 - \frac{1}{2}\epsilon_{jlm} dy^l \wedge dy^m \wedge dx^i\,, \qquad 1 \le i,j \le 3\,, \\
\beta_0 &= dy^1 \wedge dy^2 \wedge dy^3\,.
\end{aligned} \tag{2.17}$$

This basis satisfies the property

$$\int_{\mathcal{M}_6} \alpha_I \wedge \beta^J = \delta_I^J\,. \tag{2.18}$$

Finally, we choose a normalization so that the holomorphic three-form Ω is

$$\Omega = dz^1 \wedge dz^2 \wedge dz^3\,. \tag{2.19}$$

One can show that

$$\Omega = \alpha_0 + \alpha_{ij}\tau^{ij} - \beta^{ij}(\operatorname{cof}\tau)_{ij} + \beta^0(\det\tau), \tag{2.20}$$

where

$$(\operatorname{cof}\tau)_{ij} \equiv (\det\tau)\tau^{-1,\mathrm{T}} = \frac{1}{2}\epsilon_{ikm}\epsilon_{jpq}\tau^{kp}\tau^{mq}\,. \tag{2.21}$$

2.5 The RR and NS flux

The flux that we turn on must be even under the Z_2 orientifold symmetry. The intrinsic parity, under $\Omega(-1)^{F_L}$, of the various fields is given in (2.14). One sees that the 3-form field strengths $F_{(3)}$, $H_{(3)}$ that are excited must be proportional to 3-forms of odd intrinsic parity. However, the quantity that must be even is the total parity, which for a p-form on the internal space is the product of this intrinsic parity and an explicit $(-1)^p$ from the reflection action on the indices [5]. Therefore, the 3-form field strengths must transform as $(F_{(3)})_{abc} \to (F_{(3)})_{abc}$, $(H_{(3)})_{abc} \to (H_{(3)})_{abc}$ under the Z_2 action. Similarly, the field strength $F_{(5)}$ must be proportional to a 5-form of even intrinsic parity. We will ensure below that the three-forms which are excited have the correct symmetry properties. The resulting 5-form field strength is then determined by the equations of motion (2.4), and automatically satisfies the correct symmetry properties.

Note that the Bianchi identities for $F_{(3)}$ and $H_{(3)}$ require that they be closed. They should thus be expressible as a linear combination of the basis vectors of $H^3(T^6, \mathbb{Z})$. All the basis elements, (2.17), are three forms of odd parity under the Z_2 action which takes $x^i, y^i \to -x^i, -y^i$. So the symmetry constraint mentioned above is automatically taken

[5] This choice of orientation is different that in [21] and is chosen to be consistent with the conventions of [5].

care of by expressing the three-forms in this manner. Finally, taking into account the quantization conditions (2.7), $F_{(3)}$ and $H_{(3)}$ can be expressed as

$$\begin{aligned}\frac{1}{(2\pi)^2\alpha'}F_{(3)} &= a^0\alpha_0 + a^{ij}\alpha_{ij} + b_{ij}\beta^{ij} + b_0\beta^0\,,\\ \frac{1}{(2\pi)^2\alpha'}H_{(3)} &= c^0\alpha_0 + c^{ij}\alpha_{ij} + d_{ij}\beta^{ij} + d_0\beta^0\,.\end{aligned} \tag{2.22}$$

Here $a^0, \alpha_{ij}, \beta^{ij}, \beta^0$ and c^0, c^{ij}, d_{ij}, d_0 are all integers. We will search for vacua maintaining the ansatz of constant fluxes (2.22) on the T^6 throughout the paper.

3. Supersymmetry

3.1 Spinor conditions

In the discussion below our conventions are as follows: The $\gamma_i, i = 0, \ldots, 9$ matrices are all real and satisfy the algebra $\{\gamma^i, \gamma^j\} = \eta^{ij}$. The matrix, γ^0, is anti-hermitian and the others are hermitian. Also,

$$\Gamma^{(4)} \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{3.1}$$

and

$$\Gamma^{(6)} \equiv i\gamma_4\gamma_5\gamma_6\gamma_7\gamma_8\gamma_9\,. \tag{3.2}$$

Both $\Gamma^{(4)}, \Gamma^{(6)}$ are hermitian with eigenvalues ± 1. For the rest we follow the conventions of [14]. Denote the spinor ϵ as

$$\epsilon = \epsilon_L + i\epsilon_R\,. \tag{3.3}$$

Here, ϵ_L is a Majorana spinor in ten dimensions. We can write

$$\epsilon_L = u \otimes \chi + u^* \otimes \chi^*\,, q \tag{3.4}$$

where * denotes complex conjugation, and $\Gamma^{(4)}u = u$, $\Gamma^{(6)}\chi = -\chi$. The complex conjugate spinors have opposite 4 and 6 dimensional helicity.

Since we are working on a T^6/Z_2 orientifold, the spinor must be invariant with respect to the Z_2 symmetry. The Z_2 action corresponds to $\Omega R_{456789}(-1)^{F_L}$, where R_{456789} stands for a reflection in the six directions. This means that

$$\epsilon_R = -\gamma_4\gamma_5\gamma_6\gamma_7\gamma_8\gamma_9\epsilon_L\,. \tag{3.5}$$

That is

$$i\epsilon_R = -\Gamma^{(6)}\epsilon_L = u \otimes \chi - u^* \otimes \chi^*\,, \tag{3.6}$$

which gives from (3.3)

$$\epsilon = 2u \otimes \chi\,. \tag{3.7}$$

So, the spinor consistent with the Z_2 orientifolding symmetry is of type B(ecker).

Now following [14] we are lead to the conditions

$$G_{(3)}\chi = 0,\ G_{(3)}\chi^* = 0\,, \qquad \text{and} \qquad G_{(3)}\gamma^{\bar{\imath}}\chi^* = 0\,, \tag{3.8}$$

where we have introduced complex coordinates such that

$$\gamma^{\bar{\imath}}\chi = 0\,. \tag{3.9}$$

The first condition in (3.8) gives

$$(G_{(3)})_{ijk} = 0\,, \qquad (G_{(3)})_{ij}^{\ \ j} = 0\,. \tag{3.10}$$

The second that:

$$(G_{(3)})_{\bar{\imath}\bar{\jmath}\bar{k}} = 0\,, \qquad (G_{(3)})_{\bar{\imath}\bar{\jmath}}^{\ \ \bar{\jmath}} = 0\,, \tag{3.11}$$

note the second condition in (3.11) kills off the $(1,2)$ terms of the kind $J \wedge d\bar{z}^a$. Finally the third condition in (3.8) gives:

$$(G_{(3)})_{\bar{\imath}\bar{\jmath}l} = 0\,. \tag{3.12}$$

Putting all this together only primitive $(2,1)$ terms in $G_{(3)}$ survive. Primitivity means that

$$J \wedge G_{(3)} = 0\,. \tag{3.13}$$

For a $(2,1)$ form this is equivalent to requiring that

$$g^{i\bar{\jmath}}(G_{(3)})_{li\bar{\jmath}} = 0\,. \tag{3.14}$$

We turn next to analyzing the requirement that $G_{(3)}$ is of $(2,1)$ type and then discuss the requirements imposed by primitivity in section 3.4.

3.2 $G_{(3)}$ of type (2,1)

Another way to phrase the condition that $G_{(3)}$ be of type (2,1) is that the (0,3), (3,0), and (1,2) terms in $G_{(3)}$ must vanish. We saw above that the moduli space of complex structures for T^6 can be parametrized by the period matrix τ^{ij}. One can show that

$$\partial_{\tau^{ij}}\Omega = k_{ij}\Omega + \chi_{ij}\,, \tag{3.15}$$

where $\chi_{ij}, 1 \leq i,j \leq 3$ are a complete set of $(2,1)$ forms. The condition that $G_{(3)}$ is of $(2,1)$ type is then equivalent to requiring that

$$\begin{aligned} \int G_{(3)} \wedge \Omega &= 0 \\ \int \bar{G}_{(3)} \wedge \Omega &= 0 \\ \int G_{(3)} \wedge \chi_{ij} &= 0\,, \qquad 1 \leq i,j \leq 3\,. \end{aligned} \tag{3.16}$$

A convenient way to impose the requirements (3.16), is by constructing the superpotential

$$W = \int G_{(3)} \wedge \Omega\,. \tag{3.17}$$

From (3.15) we find that

$$\partial_{\tau^{ij}} W = k_{ij} W + \int G \wedge \chi_{ij}\,. \tag{3.18}$$

Similarly,

$$\partial_\phi W = -H \wedge \Omega = \frac{1}{(\phi - \bar{\phi})} \int (G_{(3)} - \bar{G}_{(3)}) \wedge \Omega \tag{3.19}$$

Thus (3.16) is equivalent to demanding that

$$W = 0 \tag{3.20a}$$

$$\partial_\phi W = 0 \tag{3.20b}$$

$$\partial_{\tau^{ij}} W = 0\,. \tag{3.20c}$$

3.3 The superpotential and equations for SUSY vacua

Using (2.22) it follows that the superpotential (3.17), is:

$$\frac{1}{(2\pi)^2\alpha'} W = (a^0 - \phi c^0) \det \tau - (a^{ij} - \phi c^{ij})(\mathrm{cof}\ \tau)_{ij} - (b_{ij} - \phi d_{ij})\tau^{ij} - (b_0 - \phi d_0)\,. \tag{3.21}$$

We see from (3.21), that it depends on ten complex variables—ϕ and the nine components of τ^{ij}. But, equations (3.20) give rise to eleven equations in these variables. Thus, generically all the equations (3.20a)–(3.20c) cannot be met and supersymmetry is broken.

The explicit equations of motion that follow from (3.20) and (3.21) are

$$a^0 \det \tau - a^{ij} (\mathrm{cof}\ \tau)_{ij} - b_{ij}\tau^{ij} - b_0 = 0\,, \tag{3.22a}$$

$$c^0 \det \tau - c^{ij} (\mathrm{cof}\ \tau)_{ij} - d_{ij}\tau^{ij} - d^0 = 0\,, \tag{3.22b}$$

$$(a^0 - \phi\, c^0)(\mathrm{cof}\ \tau)_{kl} - (a^{ij} - \phi\, c^{ij})\epsilon_{ikm}\epsilon_{jln}\tau^{mn} - (b_{ij} - \phi\, d_{ij})\delta^i_k \delta^j_l = 0\,. \tag{3.22c}$$

Here, the first equation comes from (3.20a) minus (3.20b), the second, from (3.20b), and the third from (3.20c).[6] The equations (3.22) are coupled non-linear equations in several variables and are difficult to solve in full generality.

It might seem odd at first glance that all nine scalars parametrizing the complex structure can be fixed, even though, as was argued in section 2.3, only six of them correspond to components of the metric and enter in the supergravity equations of motion. This happens because the requirements for $\mathcal{N} = 1$ supersymmetric solutions are stronger than the requirements which would follow from searching for generic solutions to the equations of motion.

3.4 Primitivity

Once the complex structure is chosen such that $G_{(3)}$ is of $(2,1)$ type, (3.14), imposes the requirement of primitivity. Note that in (3.14) the index l can take values $\{1,2,3\}$, so primitivity gives rise to three complex equations or equivalently six real equations. The space of Kähler forms is 9 dimensional to begin with so generically this will leave a three dimensional moduli space of Kähler deformations.[7]

[6] In deriving the third equation, it is useful to note the relations $\det \tau = \frac{1}{3}\epsilon_{ikm}\epsilon_{jln}\tau^{ij}\tau^{kl}\tau^{mn}$, and $(\mathrm{cof}\ \tau)_{ij} = \frac{1}{2}\epsilon_{ikm}\epsilon_{jln}\tau^{kl}\tau^{mn}$.

[7] The surviving Kähler moduli have axionic partners which come from the C_4 field, together these give rise to three chiral superfields at low energies. The six Kähler moduli which get heavy also have partners, these obtain a mass due to Chern-Simons couplings (2.1), (2.3).

Equation (3.14) can be thought of as 6 linear equations in the 9 metric components $g^{i\bar{j}}$. Solving these is relatively straightforward. In contrast we saw above that requiring G to be of type $(2,1)$ results in coupled non-linear equations which are considerably harder to work with. In practice, in the examples below, it will sometimes be easier to ensure primitivity by directly imposing the condition (3.13) on the Kähler two-form.

It is worth making one more comment at this stage. We mentioned in section 2.1 that the equations of motion can be solved if $G_{(3)}$ is an imaginary self-dual three form. This allows $G_{(3)}$ to be of three types: primitive $(2,1)$, $(0,3)$, or $(1,2)$ of the kind $J \wedge d\bar{z}^a$. We also saw in section 2.2 that in all these cases, the scalar potential for the moduli was minimized and equal to zero. Supersymmetry on the other hand is preserved if $G_{(3)}$ is purely a primitive $(2,1)$ form. Thus for choices of complex structure and Kähler class where $G_{(3)}$ has $(0,3)$ or $(1,2)$ terms, some auxilary F or D term must get a vev. However, since the potential continues to vanish in these cases, these F- and D-terms cannot be present in the scalar potential. Part of this discussion is already familiar from the study of a generic Calabi Yau manifold [5]. If $G_{(3)}$ has a $(0,3)$ term the F-component of the volume modulus gets a vacuum expectation value, however this F-component does not enter the potential because of the no-scale structure of the four-dimensional supergravity theory. Similarly when $(1,2)$ terms are present auxiliary D-terms must acquire expectation values in general. The absence of these terms in the potential can probably best be understood in the context of the underlying $\mathcal{N}=4$ supersymmetry present in the T^6/Z_2 case. We leave a more systematic analysis of the low-energy supergravity theory along the lines of [22, 23, 20] for future work; such analyses for the case of generic Calabi-Yau threefolds with fluxes have appeared in e.g. [2, 24, 25].

4. Some supersymmetric solutions

The equations which determine the value of the moduli are dificult to solve in general. The main challenge are the coupled non-linear equations (3.22) which determine the complex structure of the torus.

We do not solve these equations in their full generality below. Instead in section 4.1 we discuss some examples, where the fluxes take simple values that allow for analytic solutions. Already these simpler cases are quite interesting. As we will see, in many cases, stable minima exist where all the complex structure moduli and some of the Kähler moduli are stabilized. Section 4.2 deals with the inverse problem: we start with some values for the moduli and ask for fluxes which stabilize the moduli at these values consistent with supersymmetry. The inverse problem is sometimes easier to solve. The solutions in section 4.1 have $\mathcal{N}=1$ supersymmetry. With a few possible exceptions this should be true of the vacua in section 4.2 as well. Section 4.3 analyses some additional cases where the fluxes lead to tractable solutions. These examples turn out to have $\mathcal{N}=3$ supersymmetry. Finally, some comments related to obtaining a general supersymmetric solution are in section 4.4.

Not all of the solutions studied in this section are physically distinct. Section 5 discusses how solutions related by $\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(6,\mathbb{Z})$ transformations should be identified. Starting with some of solutions found in this section, other physically distinct solutions can be obtained by rescaling the fluxes, or carrying out $\mathrm{GL}(2,\mathbb{Z}) \times \mathrm{GL}(6,\mathbb{Z})$ transformations. This is illustrated in some examples here and discussed more fully in section 6.

One final comment before turning to examples. One would like to know if the analysis of $\mathcal{N}=1$ supersymmetric vacua in this section, receives significant α' and g_s corrections. We have not discussed an explicit $\mathcal{N}=1$ superspace description of the the low-energy effective theory in the presence of fluxes in this paper. But it is clear that such a description would involve both a superpotential (3.17), and D-terms.[8] The superpotential must be exact in the α' expansion since the partner of volume modulus is an axion which cannot occur in the α' (or string loop) corrections to the superpotential. Quite plausibly, in this case, this is true of the D terms as well, since they are related by the underlying $\mathcal{N}=4$ symmetry to the F-terms. The dilaton in the examples below is typically stabilized at a value of order one. One can be hopeful that the resulting g_s corrections (e.g. to the Kähler potential of the low-energy field theory) do not qualitatively alter our conclusions, at least in some of the examples studied here.

4.1 Example 1: fluxes proportional to the identity

We begin by studying the case where,

$$(a^{ij}, b_{ij}, c^{ij}, d_{ij}) = (a, b, c, d)\, \delta_{ij}\,, \tag{4.1}$$

that is all the flux matrices are diagonal and proportional to the identity.

The equations determining the complex structure, (3.22) will be considered first, followed by the conditions for primitivity.

With the flux matrices of the form (4.1), it is easy to see from (3.22), that the period matrix must be diagonal,

$$\tau^{ij} = \tau\delta^{ij}\,. \tag{4.2}$$

(In fact this is more generally true if the flux matrices are all diagonal).

The equations of motion (3.22) then take the form

$$P_1(\tau) \equiv a^0\tau^3 - 3a\tau^2 - 3b\tau - b_0 = 0\,, \tag{4.3}$$

$$P_2(\tau) \equiv c^0\tau^3 - 3c\tau^2 - 3d\tau - d_0 = 0\,, \tag{4.4}$$

$$(a^0 - \phi\, c^0)\tau^2 - 2(a - \phi\, c)\tau - (b - \phi\, d) = 0\,. \tag{4.5}$$

We are only interested in solutions in which τ is complex (since solutions with $\mathrm{Im}(\tau) = 0$ lie at boundaries of the moduli space). It is straightforward to show that in this case,[9]

$$P_1(\tau) = (f\tau + g)P(\tau)\,, \qquad P_2(\tau) = (h\tau + k)P(\tau)\,, \tag{4.6a}$$

[8] These play a role in ensuring primitivity of $G_{(3)}$ for example.

[9] P_1 and P_2 are cubic polynomials with real coefficients, that share a common complex root, τ. Therefore, $\bar{\tau}$ is also a root, and the two equations share a common quadratic factor. This common factor is proportional to $P = c^0 P_1 - a^0 P_2$, which has integer coefficients. Since P_1 and P_2 also have integer coefficients, it follows that P_1/P and P_2/P are each binomials with rational coefficients. But, a polynomial with integer coefficients that factorizes over the rationals also factorizes over the integers.

for some

$$P(\tau) = l\tau^2 + m\tau + n\,, \qquad f, g, h, k, l, m, n \in \mathbb{Z}\,. \tag{4.6b}$$

Thus, τ is a root of $P(\tau)$ and ϕ is determined from equation (4.5). Note that not every septuple (f, g, h, k, l, m, n) corresponds to integral flux. From the relations

$$\begin{aligned} fm + gl &= -3a\,, \qquad & fn + gm &= -3b\,, \\ hm + kl &= -3c\,, \qquad & hn + km &= -3d\,, \end{aligned} \tag{4.7}$$

we have consistency conditions modulo 3.

The D3-brane charge of the flux in this solution is given by

$$\begin{aligned} N_{\text{flux}} = \frac{1}{(2\pi)^4(\alpha')^2}\int H_{(3)} \wedge F_{(3)} &= \left(b_0 c^0 - a^0 d_0\right) + 3(bc - ad) \\ &= -\frac{1}{3}(fk - gh)(m^2 - 4ln)\,, \end{aligned} \tag{4.8}$$

which has the property that it is always $0 \pmod 3$.[10] One can also show that the result (4.8) is explicitly positive in our conventions.[11]

In summary, starting with fluxes of the form (4.1), the neccessary and sufficient condition for a non-singular solution, is the existence of integers (f, g, h, k, l, m, n) which satisfy the conditions, (4.7), and which give rise to nonzero three brane charge, (4.8).

In practice, determining polynomials of the form (4.6), by direct scrutiny is often easier than finding appropriate septuples (f, g, h, k, l, m, n).

As a concrete example, consider the case

$$P_1(\tau) \equiv \tau^3 - 1 = 0 \tag{4.9}$$

$$P_2(\tau) \equiv \tau^3 + 3\tau^2 + 3\tau + 2 = 0 \tag{4.10}$$

Both polynomials share a common factor $P(\tau) = \tau^2 + \tau + 1$ and can be expressed as:

$$P_1 \equiv (\tau - 1)P(\tau) = 0 \tag{4.11}$$

$$P_2 \equiv (\tau + 2)P(\tau) = 0\,. \tag{4.12}$$

Solving $P(\tau) = 0$ with the condition $\mathrm{Im}(\tau) > 0$, gives

$$\tau = e^{\frac{2\pi i}{3}}\,. \tag{4.13}$$

ϕ is obtained from (4.5), and given by

$$\phi = \tau = e^{\frac{2\pi i}{3}}\,. \tag{4.14}$$

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[10] To see this, note that (4.7) can be written as $\left(\begin{smallmatrix} f & g \\ h & k \end{smallmatrix}\right)\left(\begin{smallmatrix} m & n \\ l & m \end{smallmatrix}\right) = -3\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Since $\left(\begin{smallmatrix} m & n \\ l & m \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} m & -2n \\ -2l & m \end{smallmatrix}\right) \pmod 3$, this means that $\left(\begin{smallmatrix} f & g \\ h & k \end{smallmatrix}\right)\left(\begin{smallmatrix} m & -2n \\ -2l & m \end{smallmatrix}\right) \equiv 0 \pmod 3$. Taking the determinant of both sides then gives $(fk - gh)(m^2 - 4ln) \equiv 0 \pmod 9$.

[11] Our conventions are $\mathrm{Im}\tau, \mathrm{Im}\phi > 0$. One can show that the factor $(fk - gh)$ in (4.8) satisfies $\mathrm{sign}(fk - gh) = \mathrm{sign}(\mathrm{Im}\phi/\mathrm{Im}\tau)$. Therefore it is positive. The other factor, $(m^2 - 4ln)$, is the discriminant of $P(\tau)$. It is negative since the roots are complex.

We see that the moduli are fixed at a very symmetric point. Since the period matrix is diagonal, the torus factorizes as $T^6 \equiv T^2 \times T^2 \times T^2$ with respect to complex structure. In fact, when viewed in F-theory, this factorization becomes $T^8 \equiv T^2 \times T^2 \times T^2 \times T^2$. Since the eigenvalues of the period matrix are all equal to one another, and to value of the dilaton-axion, all the four 2-tori have the same modular parameter.

From (4.11), (4.12), we see that the septuple

$$(f, g, h, k, l, m, n) = (1, -1, 1, 2, 1, 1, 1)\,. \tag{4.15}$$

Also from (4.9), (4.10), and (4.3), (4.4), we see that the integers

$$(a^0, a, b, b_0) = (1, 0, 0, 1) \qquad (c^0, c, d, d_0) = (1, -1, -1, -2) \tag{4.16}$$

Either way, we find that the three-brane charge carried by the flux is given by

$$N_{\text{flux}} = \frac{1}{(2\pi)^4(\alpha')^2} \int H_{(3)} \wedge F_{(3)} = 3\,. \tag{4.17}$$

Notice that most of the non-zero fluxes in (4.16) are odd integer. We discussed in Section 2.1 why consistency on the T^6/Z_2 orientifold requires additional discrete flux to be turned on when odd integer flux is present.

To avoid this complication we can simply choose the fluxes to be twice the values indicated in (4.16). That is

$$(a^0, a, b, b_0) = (2, 0, 0, 2) \qquad (c^0, c, d, d_0) = (2, -2, -2, -4)\,, \tag{4.18}$$

and,

$$(f, g, h, k, l, m, n) = (2, -2, 2, 4, 1, 1, 1)\,. \tag{4.19}$$

No discrete flux in needed now. Since doubling all fluxes simply rescales the superpotential by an overall factor, the equations determining the moduli (3.22), remain the same and therefore the solutions for the moduli are still given by (4.13), (4.14).

From (4.8), we see that after doubling the fluxes

$$N_{\text{flux}} = 12\,. \tag{4.20}$$

Eq. (2.6), now implies that for a consistent solution we need to add ten wandering branes in addition, i.e., $N_{D3} = 10$.

This completes our discussion of how the complex structure moduli are determined, in this case. To complete our analysis we must next impose the requirement that the three flux $G_{(3)}$ is primitive. Before doing so though, let us pause to make two comments.

First, other closely related examples can be obtained by starting with the fluxes (4.16), and doing other rescalings. For example, one can double the $H_{(3)}$ flux while increasing the $F_{(3)}$ flux by a factor of four so that the resulting values for the fluxes are:

$$(a^0, a, b, b_0) = (4, 0, 0, 4) \qquad (c^0, c, d, d_0) = (2, -2, -2, -4)\,. \tag{4.21}$$

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Now, it is straightforward to see from (4.3), (4.4), that the resulting value for τ, ϕ, are:

$$\tau = e^{\frac{2\pi i}{3}}, \qquad \phi = 2e^{\frac{2\pi i}{3}}. \tag{4.22}$$

The resulting contribution to three brane charge is given by:

$$N_{\text{flux}} = 24\,, \tag{4.23}$$

so that the $N_{D3} = 4$. The rescalings discussed in (4.21), illustrate a more general feature which will be dealt with in more generality in section 6: given a solution, additional ones can be obtained by carrying out $\mathrm{GL}(6,\mathbb{Z}) \times \mathrm{GL}(2,\mathbb{Z})$ transformations on the fluxes and the moduli, provided the resulting contribution to D3-brane charge is within bounds.

Second, one would like to know whether there are other solutions with fluxes of the form, (4.1), which are not related to those discussed above by $\mathrm{GL}(6,\mathbb{Z}) \times \mathrm{GL}(2,\mathbb{Z})$ transformations or rescalings. While we do not give all the details here, it is straightforward to tabulate all choices of fluxes (or equivalently choices of the septuple (f, g, h, k, l, m, n)) which meet the requirements for the existence of $\mathcal{N} = 1$ supersymmetric solutions. In all these cases one finds that the resulting values for the moduli are related to (4.13), (4.14), by a rescaling or $\mathrm{GL}(6,\mathbb{Z}) \times \mathrm{GL}(2,\mathbb{Z})$ transformations. We have not studied the corresponding fluxes exhaustively, but in several cases they too are related to (4.16), by the same rescaling or $\mathrm{GL}(6,\mathbb{Z}) \times \mathrm{GL}(2,\mathbb{Z})$ transformation.

Primitivity. We must also verify that (at least on some subspace of the Kähler moduli space), the $G_{(3)}$ flux found from the superpotential above is primitive. We will go through this for the flux in example 1. A similar analysis (without substantially more complexity) would apply to our other examples.

In the case at hand, the flux takes the form (4.1). More explicitly,

$$\begin{aligned}
F &= a^0 dx^1 \wedge dx^2 \wedge dx^3 + a\big(dx^1 \wedge dx^2 \wedge dy^3 + \text{cyc. perms of } 123\big) - \\
&\quad - b\big(dx^1 \wedge dy^2 \wedge dy^3 + \text{ cyc. perms of } 123\big) + b_0 dy^1 \wedge dy^2 \wedge dy^3\,, \\
H &= c^0 dx^1 \wedge dx^2 \wedge dx^3 + c\big(dx^1 \wedge dx^2 \wedge dy^3 + \text{ cyc. perms of } 123\big) - \\
&\quad - d\big(dx^1 \wedge dy^2 \wedge dy^3 + \text{ cyc. perms of } 123\big) + d_0 dy^1 \wedge dy^2 \wedge dy^3\,.
\end{aligned} \tag{4.24}$$

In the present example, it is convenient to impose the requirement of primitivity in the form of (3.13),

$$J \wedge G_{(3)} = 0\,. \tag{4.25}$$

We are interested in the subspace of Kähler forms for which this requirement is met.

Take J to be of the form

$$J = \sum_{a=1}^{3} r_a^2 \; dz^a \wedge d\bar{z}^a \sim \sum_{a=1}^{3} i r_a^2 \; dx_a \wedge dy_a \tag{4.26}$$

where the second expression uses the fact that the complex structure τ of all the three T^2s, as given in (4.13), are equal. Now, notice that each term in F and H as given in (4.24) contains no repeat superscripts: one either chooses dx^a or dy^a for each of $a = 1, 2, 3$, and then wedges the three one-forms together. But the Kähler form in (4.26) contains a sum of two-forms, each of which looks like $dx^a \wedge dy^a$. The wedge product of each such term with $G_{(3)}$ will clearly vanish, because it hits either another dx^a or another dy^a in each term in F and H. Therefore, $J \wedge G_{(3)} = 0$ for the most general J of the form (4.26).

Is there a larger subspace of Kähler moduli space that preserves the primitivity? Since G is of type (2,1) and J is a (1,1) form, $J \wedge G$ is a (3,2) form. There are three nontrivial (3,2) forms on the T^6, so we expect that requiring $J \wedge G = 0$ will yield three nontrivial complex equations. The space of Kähler forms has real dimension 9, so generically we expect only a three-dimensional subspace of the Kähler moduli space (suitably complexified by the addition of axions in the relevant chiral multiplets) to parametrize flat directions of this $\mathcal{N} = 1$ theory. However, in the case at hand, the $G_{(3)}$ flux is particularly simple and non-generic, and the number of flat directions parametrized by Kähler moduli is 6 instead of 3. One can see the three "extra" flat directions by inspection. For instance, consider the two-form

$$\omega \sim i(dx^1 \wedge dy^2 + dx^2 \wedge dy^1) \tag{4.27}$$

One can easily check from (4.24) that $\omega \wedge G = 0$. Further, since the complex structure of all three T^2's is the same, it is easy to check that

$$\omega \sim dz^1 \wedge d\bar{z}^2 + dz^2 \wedge d\bar{z}^1 \,, \tag{4.28}$$

so that ω is of type $(1,1)$. Analogous perturbations with $\{1,2\}$ replaced by $\{1,3\}$ and $\{2,3\}$ similarly maintain the primitivity of $G_{(3)}$. So the $\mathcal{N} = 1$ vacua persist along a six-dimensional slice of the Kähler moduli space.

One final comment is in order. Our analysis has ensured that the solutions discussed above have at least $\mathcal{N} = 1$ supersymmetry, but it does not preclude the possibility of enhanced supersymmetry. A simple check is the following: enhanced supersymmetry requires that additional choices of complex structure are possible, in which $G_{(3)}$ is still of the kind $(2,1)$ (and primitive). $\mathcal{N} = 2$ and $\mathcal{N} = 3$ require one and two additional choices of complex structure respectively. In the solutions above, with $T^6 \equiv T^2 \times T^2 \times T^2$, there is a complete permutation symmetry among the three two-tori. This ensures that, upto an overall constant, $G_{(3)}$ must have the form,

$$G_{(3)} \sim (dz^1 \wedge dz^2 \wedge d\bar{z}^3 + dz^2 \wedge dz^3 \wedge d\bar{z}^1 + dz^3 \wedge dz^1 \wedge d\bar{z}^2) \,. \tag{4.29}$$

Other choices of complex structure can be made, by taking $z^i \rightarrow \bar{z}^i$ for some or all of the three T^2's, but none of them preserve the $(2,1)$ nature of $G_{(3)}$. So we see that these examples have only $\mathcal{N} = 1$ supersymmetry. A detailed examination of the conditions for $\mathcal{N} = 2$ supersymmetry is presented in section 7, and some more comments on this matter can be found there.

4.2 The inverse problem: fluxes from moduli

In the previous section we started with some fluxes and asked what are the resulting values for moduli in an $\mathcal{N} = 1$ susy vacuum. In this section we address the inverse problem, namely: start with some values for the moduli and ask if there are fluxes which can be turned on such that the resulting potential stabilizes the moduli at the values we begin with, while preserving $\mathcal{N}=1$ susy. The inverse problem is sometimes easier to solve and helpful in understanding the full set of consistent vacuua.

Our discussion will not be exhaustive. Instead we will consider one illustrative case. In section 4.1 we started with flux matrices which were all proportional to the identity (4.1), then argued that the period matrix must be a multiple of the identity. Here, we start by fixing the period matrix to be a multiple of the identity:

$$\tau^{ij} = \mathrm{diag}(\tau,\tau,\tau)\,, \tag{4.30}$$

then ask what values of the fluxes can yield such a solution while preserving $\mathcal{N}=1$ supersymmetry. Our notation in this section will be chosen to be consistent with section 4.1.

We begin by writing

$$a^{ij} = a\delta^{ij} + \widetilde{a}^{ij}\,, \qquad \mathrm{tr}\,\widetilde{a} = 0\,, \tag{4.31}$$

with similar relations for b, c, d. Equations (3.22a) and (3.22b) then become

$$a^0\tau^3 - 3a\tau^2 - 3b\tau - b_0 = 0\,, \tag{4.32}$$

$$c^0\tau^3 - 3c\tau^2 - 3d\tau - d_0 = 0\,, \tag{4.33}$$

and $\partial_{\tau^{ij}} W = 0$ becomes

$$\begin{aligned} (a^0 - \phi\, c^0)\tau^2 - 2(a - \phi\, c)\tau - (b - \phi\, d) &= 0\,, \\ (\widetilde{a}^{ji} - \phi\, \widetilde{c}^{ji})\tau - (\widetilde{b}_{ij} - \phi\, \widetilde{d}_{ij}) &= 0\,. \end{aligned} \tag{4.34}$$

Equation (4.34) arises by taking the trace and traceless parts of the third equation in (3.22). It can be be summarized as

$$\phi = \frac{a^0\tau^2 - 2a\tau - b}{c^0\tau^2 - 2c\tau - d} = \frac{\widetilde{a}^{ji}\tau - \widetilde{b}_{ij}}{\widetilde{c}^{ji}\tau - \widetilde{d}_{ij}}\,. \tag{4.35}$$

In the notation of (4.3), (4.4), and (4.6), the first expression for ϕ in (4.35) is

$$\frac{P_1(\tau)}{P_2(\tau)} = \frac{\big((f\tau + g)P(\tau)\big)'}{\big((h\tau + k)P(\tau)\big)'}\,, \tag{4.36}$$

where a prime denotes differentiation with respect to τ. At $P(\tau) = 0$, this reduces to $(f\tau + g)/(h\tau + k)$ and (4.35) becomes

$$\phi = \frac{f\tau + g}{h\tau + k} = \frac{\widetilde{a}^{ji}\tau - \widetilde{b}_{ij}}{\widetilde{c}^{ji}\tau - \widetilde{d}_{ij}}\,. \tag{4.37}$$

So, given a solution with $a^{ij}, b_{ij}, c^{ij}, d_{ij}$ proportional to the identity, we can generate a new solution with the same τ by, for example, turning on

$$\widetilde{a}^{ji} = f n_{ij}\,, \qquad \widetilde{b}^{ji} = -g n_{ij}\,, \qquad \widetilde{c}^{ij} = h n_{ij}\,, \qquad \widetilde{d}^{ij} = -k n_{ij}\,, \tag{4.38}$$

with n_{ij} an arbitrary traceless integer-valued matrix. This is still not the most general solution. For each i,j, equation (4.37) is two real equations in the four integers $\widetilde{a}^{ji}, \widetilde{b}_{ij}, \widetilde{c}^{ji}, \widetilde{d}_{ij}$, for which we have found a $\mathbb{Z}$'s worth of solutions parametrized by $n_{ij} \in \mathbb{Z}$. More complicated solutions will fill out a $\mathbb{Z}^2$'s worth for each i,j. In addition, the requirement that, for example, a and $\widetilde{a}^{ij}$ each be integer valued is too strict. We really only require $a\delta^{ij} + \widetilde{a}^{ij} = a^{ij}$ to be integer valued, and similarly for b, c, d.

Finally, the D3-brane charge from flux in this solution can be shown to generalize from (4.8) to

$$N_{\text{flux}} = \frac{1}{(2\pi)^4(\alpha')^2}\int H_{(3)} \wedge F_{(3)} = -\frac{1}{3}\left(1 + \frac{1}{3}\sum_{ij} {n_{ij}}^2\right)(fk - gh)(m^2 - 4ln)\,. \quad (4.39)$$

As a concrete example consider starting with the values:

$$(a^0, a, b, b_0) = (2, 0, 0, 2)\,, \qquad (c^0, c, d, d_0) = (2, -2, -2, 4)\,, \quad (4.40)$$

which were considered in (4.18), of section 4.1. In this case,

$$(f, g, h, k, l, m, n) = (2, -2, 2, 4, 1, 1, 1)\,. \quad (4.41)$$

Since (4.32) and (4.33), are the same as (4.3) and (4.4), τ is given by (4.13). Also, since the first equation in (4.34), is the same as (4.5), ϕ is given by (4.14).

The D3-brane charge is given by, (4.39),

$$N_{\text{flux}} = 12 + 4\sum_{ij} n_{ij}^2\,. \quad (4.42)$$

Now it is easy to find many non-diagonal matrices where $\sum_{ij} n_{ij}^2 = 1, 2, 3, 4, 5$. Each of them gives a consistent solution, with N_{flux} taking values, $N_{\text{flux}} = 12, 16, 20, 24, 28, 32$ respectively. Also, we should point out that since, (f, g, h, k) are even (4.41), the resulting values of $\widetilde{a}^{ij}, \widetilde{b}_{ij}, \widetilde{c}^{ij}, \widetilde{d}_{ij}$ are all even as well, (4.38), and thus all the fluxes are even.

One last comment. We argued towards the end of the previous section 4.1 that the examples discussed in it had $\mathcal{N} = 1$ supersymmetry, and no more. The examples in this section are closely related to those in section 4.1, and we expect that they too will generically have only $\mathcal{N} = 1$ supersymmetry.

4.3 More general fluxes

Section 4.1, discussed the case where the flux matrices $(a^{ij}, b_{ij}, c^{ij}, d_{ij})$ are proportional to the identity matrix. Here we would like to consider flux matrices which are diagonal but with unequal eigenvalues. In these cases one can still argue that the period matrix is diagonal, $\tau^{ij} = \text{diag}(\tau_1, \tau_2, \tau_3)$. As viewed from F-theory then, the resulting compactification is a product of four two-tori, but the modular parameters are in general unequal. Unfortunately, solving the equations for the most general set of diagonal flux matrices is a difficult task.

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To proceed we need to place additional restrictions on the flux matrices. Let us begin by considering fluxes of the form:

$$\begin{aligned} a^{ij} &= \mathrm{diag}(a_1, a_2, a_2)\,, \qquad & c^{ij} &= \mathrm{diag}(a^0, c, c)\,, \\ b_{ij} &= \mathrm{diag}(b_1, b_2, b_2)\,, \qquad & d_{ij} &= -\,\mathrm{diag}(d_1, a_2, a_2)\,, \qquad d_0 = -b_1\,. \end{aligned} \tag{4.43}$$

Setting $\tau^{ij} = \mathrm{diag}(\tau_1, \tau_2, \tau_3)$, the superpotential (3.21), is now given by

$$\begin{aligned} W = & -c^0 \phi \tau_1 \tau_2 \tau_3 + a^0 (\tau_1 + \phi) \tau_2 \tau_3 + c(\tau_2 + \tau_3) \phi \tau_1 - \\ & - a_1 \tau_2 \tau_3 - d_1 \phi \tau_1 - a_2 (\tau_1 + \phi)(\tau_2 + \tau_3) - \\ & - b_1 (\phi + \tau_1) - b_2 (\tau_2 + \tau_3) - b_0 \end{aligned} \tag{4.44}$$

One can see that the superpotential is symmetric between $\phi \leftrightarrow \tau_1$ and $\tau_2 \leftrightarrow \tau_3$. Thus, for the restricted choice, (4.43), one can consistently seek solutions where the four modular parameters take at most two distinct values.

We now turn to describing two examples where additional restrictions lead to tractable solutions.

Example 2. In the first example, we set all trilinear and linear terms in the superpotential, (4.44), to be zero, i.e.,

$$a^0 = c = b_1 = b_2 = 0\,. \tag{4.45}$$

In this case the superpotential takes the form

$$W = -c^0 \phi \tau_1 \tau_2 \tau_3 - a_1 \tau_2 \tau_3 - d_1 \phi \tau_1 - a_2 (\tau_1 + \phi)(\tau_2 + \tau_3) - b_0\,. \tag{4.46}$$

Setting $\partial_\phi W = 0, \partial_{\tau_i} W = 0$ shows that

$$\tau_1 = \phi\,, \qquad \tau_2 = \tau_3\,, \tag{4.47}$$

(as expected) and in addtion leads to two equations:

$$-c^0 \tau_1 \tau_2^2 - d_1 \tau_1 - 2a_2 \tau_2 = 0\,, \tag{4.48}$$

$$-c^0 \tau_1^2 \tau_2 - a_1 \tau_2 - 2a_2 \tau_1 = 0\,, \tag{4.49}$$

where in both equations we have substituted for ϕ, τ_3, using (4.47).

These lead to the relation,

$$\tau_2^2 = \frac{d_1}{a_1} \tau_1^2\,, \tag{4.50}$$

i.e.,

$$\tau_2 = \pm \sqrt{\frac{d_1}{a_1}} \tau_1\,. \tag{4.51}$$

Substituting in (4.48) gives

$$\tau_1 = i \sqrt{\frac{a_1}{d_1 c^0}} \left(d_1 \pm 2a_2 \sqrt{\frac{d_1}{a_1}} \right)^{1/2}. \tag{4.52}$$

Setting $W = 0$ then leads to a condition determining b_0 in terms of the other flux integers,

$$b_0 = \frac{a_1}{d_1 c^0}\left(d_1 \pm 2a_2\sqrt{\frac{d_1}{a_1}}\right)^2 . \tag{4.53}$$

Finally, the contribution to the three brane charge is then given by

$$N_{\rm flux} = 2a_1 d_1 + 6a_2^2 \pm 4a_2\sqrt{a_1 d_1} . \tag{4.54}$$

To find a consistent non-singular solution we need to choose integers c^0, a_1, d_1 and a_2 such that τ_1, τ_2 are complex, b_0 is an integer, and the total flux $N_{\rm flux}$ is within bounds.

One solution to these conditions is obtained by taking

$$(a_1, d_1, a_2, c^0) = (1, 1, -1, -1) , \tag{4.55}$$

and choosing the positive sign in (4.51), so that

$$\tau_2 = +\sqrt{\frac{d_1}{a_1}}\tau_1 = \tau_1 . \tag{4.56}$$

Then from (4.52), we find

$$\tau_1 = i \tag{4.57}$$

and from (4.54),

$$N_{\rm flux} = 4 . \tag{4.58}$$

Also, from (4.53), $b_0 = -1$ and is indeed an integer.

Notice that the integers (4.55) are odd. As in example 1, section 4.1, to avoid complications related to adding discrete flux we can obtain a consistent solution by doubling all the fluxes so that

$$(a_1, d_1, a_2, c^0, b^0) = (2, 2, -2, -2, -2) . \tag{4.59}$$

The modular parameters are unchanged and given by (4.57), (4.56), (4.47). The total flux is

$$N_{\rm flux} = 16 , \tag{4.60}$$

which means 8 dynamical D3-branes need to be added for a consistent solution.

It turns out that the solution above has $\mathcal{N} = 3$ supersymmetry. Vacua with $\mathcal{N} = 3$ are analysed in generality in the recent paper [15]. The possibility of $\mathcal{N} = 3$ supersymmetry was also mentioned in [4]. The solution above is in fact a special case of the examples found in [15]. To see that it has $\mathcal{N} = 3$ supersymmetry, we note that with the flux (4.59) and the moduli, $\phi = \tau^i = i$, $G_{(3)}$ takes the form

$$\frac{1}{(2\pi)^2\alpha'} G_{(3)} = 2i d\bar{z}^1 \wedge dz^2 \wedge dz^3 . \tag{4.61}$$

It then follows that two additional complex structures in which $G_{(3)}$ is still of type (2,1) can be defined by taking the complex coordinates on the three T^2's to be $(w^1, w^2, w^3) = (\bar{z}^1, \bar{z}^2, z^3)$ or $(w^1, w^2, w^3) = (\bar{z}^1, z^2, \bar{z}^3)$. Thus, as per the general discussion in [15] (see also section 4.1 above), the solution has $\mathcal{N} = 3$ supersymmetry.

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Let us also add that additional solutions can be obtained by starting with the (4.55), (4.56), (4.57), and doing $\mathrm{GL}(6,\mathbb{Z})\times\mathrm{GL}(2,\mathbb{Z})$ transformations. In particular one can obtain a solution in which $N_{\text{flux}} = 32$, as will be discussed in more detail in the examples of section 6.

Example 3. In the next example we again start with flux matrices and superpotential of the form (4.43), (4.44), respectively, but now set the following additional restrictions on the fluxes:

$$c^0 = 0\,, \qquad c = -a^0\,, \qquad a_2 = 0\,, \qquad d_1 = -a_1\,, \qquad b_2 = -b_1\,. \tag{4.62}$$

The superpotential (4.44) then takes the form

$$\begin{aligned} W = &+a^0(\tau_1+\phi)\tau_2\tau_3 - a^0(\tau_2+\tau_3)\phi\tau_1 - a_1\tau_2\tau_3 + \\ &+a_1\phi\tau_1 - b_1(\phi+\tau_1) + b_1(\tau_2+\tau_3) - b_0\,. \end{aligned} \tag{4.63}$$

Solving the equations $\partial_\phi W = 0, \partial_{\tau_i} W = 0$, it is easy to see that

$$\tau_1 = \phi = \tau_2 = \tau_3 \equiv \tau\,, \tag{4.64}$$

with τ given by

$$\tau = \frac{a_1 \pm \sqrt{a_1^2 - 4a^0 b_1}}{2a^0}\,, \tag{4.65}$$

is a solution. Setting $W = 0$ yields the condition that

$$b_0 = 0\,. \tag{4.66}$$

Finally the D3-brane charge contribution is

$$N_{\text{flux}} = 4b_1 a^0 - a_1^2\,. \tag{4.67}$$

Consistent solutions can be found by taking

$$a^0 = 2\,, \qquad b_1 = 2\,, \qquad a_1 = 2\,. \tag{4.68}$$

This yields

$$\tau = \frac{1 \pm i\sqrt{3}}{2} \tag{4.69}$$

and

$$N_{\text{flux}} = 12\,. \tag{4.70}$$

Alternatively, one can take

$$a^0 = 2\,, \qquad b_1 = 4\,, \qquad a_1 = 2\,. \tag{4.71}$$

In this case,

$$\tau = \frac{1 \pm i\sqrt{7}}{2} \tag{4.72}$$

and

$$N_{\text{flux}} = 28\,. \tag{4.73}$$

Note that unlike example 2 above, the two-tori in (4.69) and (4.72) are not square.

Once again, doing general rescalings and $\mathrm{GL}(6,\mathbb{Z}) \times \mathrm{GL}(2,\mathbb{Z})$ transformations leads to additional solutions in each of these cases.

As in the previous example, the solutions discussed here have $\mathcal{N} = 3$ supersymmetry as well. This follows by the same argument as in the previous example, after noting that in both the cases (4.68) and (4.71), $G_{(3)}$ can be expressed as

$$\frac{1}{(2\pi)^2\alpha'} G_{(3)} = a^0 (d\bar{z}^1 \wedge dz^2 \wedge dz^3)\,. \tag{4.74}$$

4.4 Toward a general supersymmetric solution

Solving the supersymmetric equations of motion (3.22) without any simplifying assumptions is a difficult task. However, a couple of observations can make the task easier. First, note that it is possible to re-write equation (3.22) as

$$\left(\mathrm{cof}\left(\tau - \frac{1}{A^0}A\right)\right)_{ij} = \frac{1}{A^{0\,2}}(\mathrm{cof}\,A)_{ij} + \frac{1}{A^0}B_{ij}\,. \tag{4.75}$$

This determines τ^{ij} in terms of the flux matrices and the dilaton ϕ, since if $\mathrm{cof}\,x = y$, then $x = \mathrm{cof}\,y/\sqrt{\det y}$.

Next, we note that one can actually eliminate the τ^{ij} from the $W = 0$ and $\partial_{\tau^{ij}} W = 0$ equations to obtain a quartic equation for ϕ. The quartic is derived in appendix B, and takes the form

$$(\det A)B_0 - (\det B)A^0 + (\mathrm{cof}\,A)_{ij}(\mathrm{cof}\,B)^{ij} + \frac{1}{4}(A^0 B_0 + A^{ij}B_{ij})^2 = 0\,, \tag{4.76}$$

where $A^0 = a^0 - \phi c^0$, $A^{ij} = a^{ij} - \phi c^{ij}$, and B_0, B_{ij} are defined similarly. A quartic equation is soluble, so one can solve (4.76) for the allowed values of ϕ.

This leaves only the equation $\partial_\phi W = 0$, which upon substitution for ϕ and τ^{ij} gives one nonlinear equation in integers. The integer equation is a consistency condition that determines whether the choice of flux can lead to a supersymmetric solution. The hard part is solving this equation. An additional complication is that for each solution to the integer equation, one must determine all consistent configurations of exotic orientifold planes (as described in [15]), if one is to find all supersymmetric solutions.

5. Distinctness of solutions

Not all solutions with different values of ϕ or τ^{ij} are physically distinct. There is an $\mathrm{SL}(2,\mathbb{Z})$ symmetry that relates equivalent values of the dilaton-axion, and an $\mathrm{SL}(6,\mathbb{Z})$ symmetry that relates equivalent values of the period matrix τ^{ij}.

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5.1 $\mathrm{SL}(2,\mathbb{Z})$ equivalence

The type-IIB supergravity action (2.1) is invariant under the $\mathrm{SL}(2,\mathbb{R})$ symmetry,

$$\begin{aligned} \begin{pmatrix} F_{(3)} \\ H_{(3)} \end{pmatrix} \to \begin{pmatrix} F'_{(3)} \\ H'_{(3)} \end{pmatrix} = m \begin{pmatrix} F_{(3)} \\ H_{(3)} \end{pmatrix}, \\ \phi \to \phi' = \frac{a\phi + b}{c\phi + d}, \qquad m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) . \end{aligned} \tag{5.1}$$

Under this symmetry, the complex 3-form flux transforms as

$$G_{(3)} \to G'_{(3)} = F'_{(3)} - \phi' H'_{(3)} , \tag{5.2}$$

which one can check is equivalent to

$$G_{(3)} \to G'_{(3)} = \frac{G_{(3)}}{c\phi + d} . \tag{5.3}$$

At the quantum level, only an $\mathrm{SL}(2,\mathbb{Z}) \subset \mathrm{SL}(2,\mathbb{R})$ survives. Solutions that differ only by $\mathrm{SL}(2,\mathbb{Z})$ transformations are equivalent. It is therefore customary to take ϕ to be in the fundamental domain $\mathcal{F}$, of the upper half plane modulo $\mathrm{PSL}(2,\mathbb{Z})$:

$$\mathcal{F} = \left\{ \phi \in \mathbb{C} \;\middle|\; \mathrm{Im}\phi > 0, -\frac{1}{2} \le \mathrm{Re}\phi \le \frac{1}{2}, |\phi| \ge 1 \right\} . \tag{5.4}$$

The examples were not chosen in such a way that the solutions would necessarily give $\phi \in \mathcal{F}$. However it is a simple matter to transform them to the fundamental domain using (5.1), where now

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) . \tag{5.5}$$

5.2 $\mathrm{SL}(6,\mathbb{Z})$ equivalence

Following [26], let

$$\mathcal{B}_{\mathbb{C}^3} = \big(\mathbf{e}_{(1)},\ \mathbf{e}_{(2)},\ \mathbf{e}_{(3)}\big) \tag{5.6}$$

denote a basis of $\mathbb{C}^3$, and consider a T^6 in which the lattice basis is

$$\begin{aligned} \mathcal{B}_{T^6} &= \big(\mathbf{e}_{(1)}, \mathbf{e}_{(2)}, \mathbf{e}_{(3)}, \mathbf{e}_{(i)}\tau^i{}_1, \mathbf{e}_{(i)}\tau^i{}_2, \mathbf{e}_{(i)}\tau^i{}_3\big) \\ &= \mathcal{B}_{\mathbb{C}^3}\Lambda , \qquad \Lambda = \big(1, \tau\big) . \end{aligned} \tag{5.7a}$$

Under a change of lattice basis,

$$\mathcal{B}_{T^6} \to \mathcal{B}'_{T^6} = \mathcal{B}_{T^6} M , \qquad M \in \mathrm{SL}(6,\mathbb{Z}) , \tag{5.8}$$

so

$$\Lambda \to \Lambda'' = \Lambda M , \qquad M \in \mathrm{SL}(6,\mathbb{Z}) . \tag{5.9}$$

The change of lattice basis does not produce Λ'' in the standard form $(1, *)$. However, under a change of $\mathbb{C}^3$ basis,

$$\Lambda'' \to \Lambda' = N\Lambda'' = N\Lambda M\,, \qquad N \in \mathrm{GL}(3,\mathbb{C})\,. \tag{5.10}$$

We can choose $N = N(M, \tau)$, so that Λ' is in standard form,

$$\Lambda' = N\Lambda M = \big(1, \tau'\big)\,. \tag{5.11}$$

Two period matrices τ and τ' related by (5.7a) and (5.11), are equivalent. Also, under an $\mathrm{SL}(6,\mathbb{Z})$ coordinate transformation M, the fluxes $F_{(3)}, H_{(3)}$, (when regarded as three-forms) must stay the same.[12] This means that two solutions with period matrices τ and τ' related by (5.7a) and (5.11), and which are otherwise identical, are equivalent.

We should make one more comment before turning to an example. In section 7 we discuss solutions which break supersymmetry. The analysis above, identifying solutions related by $\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(6,\mathbb{Z})$ transformations, applies to these cases as well.

5.3 Example

To illustrate the equivalences, consider example 1 from section 4. Suppose instead of choosing the two polynomials (4.9), (4.10), we made the following choices:

$$P_1(\tau) \equiv -(\tau^3 + 1) = 0\,, \tag{5.13}$$

$$P_2(\tau) \equiv 2\tau^3 - 3\tau^2 + 3\tau - 1 = 0\,. \tag{5.14}$$

These two polynomials have a common factor $P(\tau) = \tau^2 - \tau + 1$, and the corresponding values of integers are

$$((a^0)', a', b', b_0') = (-1, 0, 0, 1) \qquad ((c^0)', c', d', d_0') = (2, 1, -1, 1)\,, \tag{5.15}$$

where the prime superscripts are being used to distinguish the present case from example 1, in section 4. Solving $P(\tau) = 0$ and choosing the solution with $\mathrm{Im}(\tau') > 0$ gives

$$\tau' = e^{i\pi/3}\,. \tag{5.16}$$

Also, solving (4.5) with (5.15) gives $\phi' = e^{i2\pi/3}$. Finally the total three brane charge in this case is $N_{\mathrm{flux}} = 3$, as follows from (4.8), (5.15).

This solution is in fact related to the one corresponding to flux, (4.16), by an $\mathrm{SL}(6,\mathbb{Z})$ transformation.

The $\mathrm{SL}(6,\mathbb{Z})$ transformation has the form, $S \otimes S \otimes S$ where, each $S \in \mathrm{SL}(2,\mathbb{Z})$, acts on the one of the three T^2's as:

$$S : \tau \to -\frac{1}{\tau}\,. \tag{5.17}$$

[12] Under the $\mathrm{SL}(6,\mathbb{Z})$ transformation, (5.8), the two coordinate systems are related as:

$$M \begin{pmatrix} x_i' \\ y_i' \end{pmatrix} = \begin{pmatrix} x_i \\ y_i \end{pmatrix}. \tag{5.12}$$

The transformation of $(F_{(3)})_{ijk}, (H_{(3)})_{ijk}$ then follow by requiring that the three forms, $F_{(3)}, H_{(3)}$ stay invariant.

To see this we note first that under (5.17), the modular parameter $\tau' = e^{i\pi/3} \to e^{2\pi i/3}$, which agrees with (4.13). Second, one can show that the corresponding matrix M, in (5.12), acting on the coordinates of each T^2 has the form $M = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. From this it follows that in order to be related by the same $\mathrm{SL}(6,\mathbb{Z})$ transformation, the flux integers, (4.16), (5.15), must satisfy the conditions:

$$((a^0)', a', b', b_0') = (-b_0, -b, a, a^0)\,, \tag{5.18}$$
$$((c^0)', c', d', d_0') = (-d_0, -d, c, c^0)\,. \tag{5.19}$$

Comparing, (4.16), and (5.15), we see that these conditions are in fact true. Finally, the two solutions have the same value for the dilaton and agree in the value for N_{flux}. Thus, as per our general discussion above, they are identical.

6. New solutions using $\mathrm{GL}(2,\mathbb{Z}) \times \mathrm{GL}(6,\mathbb{Z})$ transformations

In various examples of section 4 we have seen that starting with a given solution, additional ones can be generated by appropriately rescaling the fluxes. Here we discuss this in more generality and show how additional solutions can be obtained by using $\mathrm{GL}(2,\mathbb{Z}) \times \mathrm{GL}(6,\mathbb{Z})$ transformations. The resulting solutions are physically distinct in general, with a different flux contribution to three brane charge. Solving the tadpole condition (2.6) without anti-branes requires that the value of N_{flux} for the new solutions is ≤ 32, and that the required number of wandering D3-branes are added in each case.

The general discussion in this section is applied to some examples at the end. These illustrate that starting with a diagonal period matrix physically distinct solutions can be obtained with a non-diagonal period matrix using the $\mathrm{GL}(\mathbb{Z})$ transformations. The examples also yield solutions where all the three brane charge is cancelled by fluxes alone, leaving in one instance, four flat directions in Kähler moduli space. These solutions are of the kind mentioned in the introduction and are good illustrations of the reduced number of moduli that survive once fluxes are turned on.

6.1 $\mathrm{GL}(2,\mathbb{Z})$ transformations

Consider a solution to the $\mathcal{N} = 1$ susy equations which has flux, $F_{(3)}$,$H_{(3)}$, and moduli fixed at values ϕ,τ^{ij}. Now transform the fluxes as follows:

$$\begin{pmatrix} F_{(3)} \\ H_{(3)} \end{pmatrix} \to \begin{pmatrix} F'_{(3)} \\ H'_{(3)} \end{pmatrix} = m \begin{pmatrix} F_{(3)} \\ H_{(3)} \end{pmatrix}, \tag{6.1}$$

where the matrix $m \in \mathrm{GL}(2,\mathbb{Z})$.[13]

One can show that a solution to the the supersymmetry conditions for the new fluxes is obtained by taking the moduli to be at the values

$$\phi' = \frac{a\phi + b}{c\phi + d}\,, \qquad (\tau^{ij})' = \tau^{ij}\,. \tag{6.2}$$

[13]By this we mean that $m = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ where $a, b, c, d \in \mathbb{Z}$. In particular $\det(m)$ need not be 1.

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To see this note that under the transformation (6.1),

$$G_{(3)} \to G'_{(3)} = \det(m) \frac{G_{(3)}}{c\phi + d} \, . \tag{6.3}$$

The resulting superpotential, (3.17), transforms to

$$W \to W'[\phi', \tau] = \int \Omega \wedge G'_{(3)} = \det(m) \frac{W[\phi, \tau]}{c\phi + d} \, . \tag{6.4}$$

where the dependence of the superpotential on the moduli has been explicitly indicated above.

It now follows that if W satisfied the supersymmetry equations, (3.20), when the moduli take values ϕ, τ^{ij}, then W' will also meet the susy eqautions for the transformed values, (6.2).

Finally, note that under the transformation of the fluxes, (6.1), the flux contribution to three brane charge becames

$$N_{\text{flux}} \to N'_{\text{flux}} = \det(m) N_{\text{flux}} \, . \tag{6.5}$$

Starting with a solution, where $N_{\text{flux}} > 0$ we are therefore restricted to $\mathrm{GL}(2, \mathbb{Z})$ transformations with $\det(m) > 0$. Also as mentioned above, we need to ensure that $N'_{\text{flux}} \leq 32$, (2.6).

6.2 $\mathrm{GL}(6, \mathbb{Z})$ transformations

Our starting point is once again a $\mathcal{N} = 1$ susy preserving solution with flux, $F_{(3)}, H_{(3)}$ and moduli fixed at values ϕ, τ^{ij}. But this time we consider transforming the flux by a $\mathrm{GL}(6, \mathbb{Z})$ transformation. The transformation can be described explicitly as follows. We fix a basis of one forms (dx^i, dy^i) as in section 2.4. The components of $F_{(3)}$ in this basis then transform as

$$(F_{(3)})_{abc} \to (F'_{(3)})_{abc} = (F_{(3)})_{def} M^d_a M^e_b M^f_c \, , \tag{6.6}$$

and similarly for $H_{(3)}$. As a result the components of $G_{(3)}$ in this basis also then transform under $\mathrm{GL}(6, \mathbb{Z})$ as:

$$(G_{(3)})_{abc} \to (G'_{(3)})_{abc} = (G_{(3)})_{def} M^d_a M^e_b M^f_c \, . \tag{6.7}$$

In (6.6), (6.7), M^a_b are the elements of a matrix, $M \in \mathrm{GL}(6, \mathbb{Z})$.

We will see that the new fluxes lead to the moduli being stabilized at values ϕ', τ' where $\phi' = \phi$ and

$$(1, \tau') = N(1, \tau) M \, . \tag{6.8}$$

In (6.8), M is the same matrix that appears in (6.7), and $N \in \mathrm{GL}(3, \mathbb{C})$ is a matrix that is chosen so that the left hand side has the form $(1, *)$. In appendix C, we show that the superpotential for the transformed flux, (6.7), is related to the original superpotential by

$$W'[\tau', \phi] = \det(N) \det(M) W[\tau, \phi] \tag{6.9}$$

where τ', τ are related as in (6.8). It then follows that if τ, ϕ solve the supersymmetry equations (3.20) for the original fluxes, τ', ϕ' are the solutions for the transformed fluxes.

Let us also note that under the transformation (6.7), the contribution to the three brane charge for the new flux is given by

$$N_{\text{flux}} \to N'_{\text{flux}} = \det(M) N_{\text{flux}} \,. \tag{6.10}$$

Once again we must ensure that the resulting value of three brane charge meets the consistency checks.

Two more comments are worth making at this stage. First, suppose the solution one began with had a diagonal period matrix τ. Then it is possible by an appropriate choice of the matrix M to obtain other solutions where the resulting period matrix τ', (6.8), is non-diagonal. A specific example will be given in the next section. Second, in the discussion above we took $M \in \mathrm{GL}(6,\mathbb{Z})$. In fact, this is not necessary. All that is required is that the transformed fluxes (6.7), have integer components in the cohomology basis (2.17).[14] For example choosing $M^a_b = \lambda\delta^a_b$, where $\lambda^3 = 2$ is perfectly acceptable. In this case, we learn from (6.8), that $N = \lambda^3 \mathbf{1}_{3\times 3}$, and $\tau' = \tau$. We have already encountered this case in section 4.1: doubling the flux rescales the superpotential and leaves the moduli fixed.

6.3 An example

For an example we start with the a solution discussed in example 2 of section 4.3. The fluxes are given by (4.59), and the resulting moduli are stabilized at $\phi = i$ and

$$\tau^{ij} = i\delta^{ij} \,, \tag{6.11}$$

eqs. (4.57), (4.56), and (4.47). The solution has $N_{\text{flux}} = 16$.

Now take the matrix M, (6.7), to be

$$M = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & D \end{pmatrix} . \tag{6.12}$$

Here $M \in \mathrm{GL}(6,\mathbb{Z})$, $\mathbf{1}$ is the 3×3 identity matrix and $D \in \mathrm{GL}(3,\mathbb{Z})$. The resulting values for the fluxes can be worked out using (6.7), but we will not do so explcitly here.

The general discussion of the previous section then tells us that the moduli are stabilized at $\phi' = \phi = i$ and τ', where τ' is given in terms of the original period matrix (6.11) as discussed in (6.8). Given M in (6.12), and τ in (6.11), it is easy to show that the matrix N in (6.8) is

$$N = \mathbf{1}_{3\times 3} \,. \tag{6.13}$$

Therefore,

$$\tau' = iD \,. \tag{6.14}$$

The flux contribution to the three brane charge in this case is given by (6.10),

$$N'_{\text{flux}} = \det(D) N_{\text{flux}} = 16 \det(D) \,. \tag{6.15}$$

[14] In fact, the coefficients should be even integers if discrete flux is not being turned on.

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Since $N'_{\text{flux}} \leq 32$, we learn that $\det(D) = 2$ is the only possibility (cases with $\det(D) = 1$ give rise to solutions related to the original one by $\mathrm{SL}(6,\mathbb{Z})$ transformations, which by the discussion in section 5.2 are not physically distinct).

As examples for D, two possibilities are

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{6.16}$$

in which case the resulting period matrix is still diagonal (6.14), but the eigenvalues are unequal. Or,

$$D = \begin{pmatrix} 1 & -3 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{6.17}$$

in which case the resulting period matrix is not diagonal. In the latter case we see that starting with a diagonal period matrix we have found an example where τ is fixed at a non-diagonal value.

It is also useful to briefly revisit the primitivity constraint in the example (6.16). Since the complex structure is diagonal, it is straightforward to verify that three Kähler deformations of the type (4.26), survive as flat directions. In addition to these deformations, the deformation with $w \sim dx^2 \wedge dy^3 + dx^3 \wedge dy^2$ is also now of type (1,1). Thus, altogether there are four Kähler flat directions. This is an example of the kind mentioned in the introduction. The fluxes contribute $N_{\text{flux}} = 32$, so no extra D3 branes are needed to soak up the orientifold three plane charge. The dilaton-axion and all complex structure moduli are lifted, leaving four surviving moduli which are Kähler deformations.

7. Solutions with $\mathcal{N} = 2$ supersymmetry.

In this section we discuss the conditions which the $G_{(3)}$ flux must satisfy to preserve $\mathcal{N} = 2$ supersymmetry. We will illustrate the discussion with one example at the end of this section. A more extensive study of $\mathcal{N} = 2$ preserving vacua is left for the future.

An $\mathcal{N} = 2$ theory has an $\mathrm{SU}(2)_R$ R-symmetry. $\mathrm{SU}(2)_R$ is embedded in $\mathrm{SO}(6)$, the group of rotations along the six dimensional compactified directions, as follows:[15]

$$\mathrm{SU}(2)_R \subset \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \subset \mathrm{SO}(4) \times \mathrm{U}(1) \subset \mathrm{SO}(6)\,. \tag{7.1}$$

We choose conventions so that the spinor representation, 4, of SO(6) transforms as a $(2,1)_{+1} + (1,2)_{-1}$ under $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_L \times \mathrm{U}(1)$, and the 6 of SO(6) as $(2,2)_0 + (1,1)_{+2} + (1,1)_{-2}$. In the discussion below we will use indices a, b to denote an element of the 6 which transforms as $(2,2)_0$ and indices l, m to denote elements transforming as $(1,1)_2, (1,1)_{-2}$.

Since $\mathrm{SU}(2)_R$ is a symmetry of the $\mathcal{N} = 2$ theory, it must be left unbroken by the compactification. This means in particular that $G_{(3)}$ must leave an SU(2) subgroup of

[15] This embedding follows by noting that the spinor ϵ, under which the dilatino and gravitino variations vanish, must be a doublet of $\mathrm{SU}(2)_R$.

SO(6) unbroken. $G_{(3)}$ transforms as $[6 \times 6 \times 6]_A$ under SO(6). With respect to $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_L \times \mathrm{U}(1)$ this decomposes as

$$[6 \times 6 \times 6]_A = (2,2)_0 + (2,2)_0 + (3,0)_2 + (3,0)_{-2} + (0,3)_2 + (0,3)_{-2} \,. \tag{7.2}$$

For $G_{(3)}$ to preserve $\mathrm{SU}(2)_R$ it can only have components along the $(0,3)_{\pm 2}$ representations. A little thought shows that this means $G_{(3)}$ has index structure $(G_{(3)})_{abl}$, in the notation introduced above.

A detailed analysis of the spinor conditions will be presented in appendix D. The conclusion is the following: in order to preserve $\mathcal{N}=2$ supersymmetry $G_{(3)}$ must only take values in the $(0,3)_2$ representation of $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_L \times \mathrm{U}(1)$. In other words, the $(0,3)_{-2}$ representation, which would have also preserved $\mathrm{SU}(2)_R$, must be absent.

Let us check that this condition on $G_{(3)}$ leads to a solution of the equations of motion. The ISD condition (2.12), can be written in the present case as

$$\epsilon_{abmcdl} G_{(3)}^{cdl} = i (G_{(3)})_{abm} \,, \tag{7.3}$$

which can also be expressed as

$$\epsilon_{abcd} \epsilon_{ml} G_{(3)}^{cdl} = i (G_{(3)})_{abm} \,. \tag{7.4}$$

The two ϵ symbols above refer to the four directions on which the SO(4) acts and the two directions on which the U(1) acts respectively. Since $G_{(3)}$ is a tensor transforming as a $(0,3)$ representation of $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_L$ it corresponds to the self dual representation of SO(4) and therefore satisfies the condition $\epsilon_{abcd}(G_{(3)})^{cdl} = G_{ab}^{l}$. Further, one can show, in our choice of conventions, that a charge 2 representation of the U(1) satisfies $\epsilon_{ml}(G_{3)})^{cdl} = i(G_{(3)})^{cd}_{m}$. From this, we see that if $G_{(3)}$ is of the $(0,3)_2$ kind, it satisfies the ISD requirement.

It is useful to relate the discussion above to that in section 3.1 where we saw that $G_{(3)}$ must be primitive and $(2,1)$ to preserve $\mathcal{N}=1$ susy. Requiring $\mathcal{N}=2$ supersymmetry must impose extra conditions on the $G_{(3)}$ flux. The requirements for $\mathcal{N}=1$ supersymmetry mean that under an $\mathrm{SU}(3) \subset \mathrm{SO}(6)$, $G_{(3)}$ transforms as a $\bar{6}$. The $\mathrm{SU}(2)_L$ discussed above is a subgroup of this SU(3) (since ϵ is a singlet under it), so the $\bar{6}$ representation of SU(3) transforms under $\mathrm{SU}(2)_L$ as $3+2+1$. As a result, we learn that $\mathcal{N}=1$ susy alone allows $G_{(3)}$ to take values in the $3+2+1$ representations of $\mathrm{SU}(2)_L$. $\mathcal{N}=2$ susy imposes the additional requirement that the doublet and singlet components are missing and $G_{(3)}$ transforms purely as a triplet under $\mathrm{SU}(2)_L$. For completeness let us also mention that for unbroken $\mathcal{N}=3$ one must impose yet a further restriction: $G_{(3)}$ must have only one non-zero component proportional to a highest weight state of the triplet representation.

These conditions can be visualised as follows. The weight diagram of the $\bar{6}$ representation is a triangle. (See, e.g., (IX.iii) of [27]). Each state in the $\bar{6}$ representation is denoted by a point in this diagram. $\mathcal{N}=3$ supersymmetry requires $G_{(3)}$ to be proportional to any one of the three vertices of the triangle, $\mathcal{N}=2$ requires that the components of $G_{(3)}$ all lie along an edge of the triangle, and finally, $\mathcal{N}=1$ supersymmetry allows components along all six points in the diagram.

In the example below it will be useful to first impose the conditions for $\mathcal{N}=1$ supersymmetry, then check if the extra restrictions for $\mathcal{N}=2$ supersymmetry are met.

7.1 An $\mathcal{N}=2$ example

As an example choose the fluxes to be:

$$\begin{aligned} H_{135} &= H_{245} = F_{136} = F_{246} = (2\pi)^2\alpha' a^0 \\ F_{135} &= F_{245} = -H_{246} = -H_{136} = (2\pi)^2\alpha' a^0 \,, \end{aligned} \tag{7.5}$$

where we are working in the coordinates x^i, y^i introduced in section 2.4. Each index above takes six possible values; $i = 1, 3, 5$, denote components along x^1, x^2, x^3 directions, $i = 2, 4, 6$, along y^1, y^2, y^3. Also in (7.5), a^0 is an integer. In the cohomology basis, (2.17), the fluxes can be expressed as

$$\frac{1}{(2\pi)^2\alpha'} F_{(3)} = a^0\alpha_0 + a^0\beta^0 - a^0\beta^{33} + a^0\alpha_{33} \tag{7.6}$$

and

$$\frac{1}{(2\pi)^2\alpha'} H_{(3)} = a^0\alpha_0 - a^0\beta^0 - a^0\beta^{33} - a^0\alpha_{33} \,. \tag{7.7}$$

The superpotential is then given by

$$W = a^0(1-\phi)\det\tau - a^0(1+\phi)(\mathrm{cof}\,\tau)_{33} + a^0(1-\phi)\tau^{33} - a^0(1+\phi) \,. \tag{7.8}$$

One can show that the equations for $\mathcal{N}=1$ supersymmetry (3.22), have the solution

$$\tau^{ij} = i\delta^{ij}, \phi = i \,. \tag{7.9}$$

The contribution to three brane flux is

$$N_{\text{flux}} = 4(a^0)^2 \,. \tag{7.10}$$

Choosing $a^0 = 2$ we have $N_{\text{flux}} = 16$ which is within the acceptable bound, (2.6).

With the choice of complex structure in (7.9), $G_{(3)}$ can now be expressed as

$$\frac{1}{(2\pi)^2\alpha'} G_{(3)} = \frac{a^0(1-i)}{2}(dz^1\wedge d\bar{z}^2\wedge dz^3 + d\bar{z}^1\wedge dz^2\wedge dz^3) \,. \tag{7.11}$$

It is clear that the primitivity condition is satisfied if one chooses the Kähler form to be of the form

$$J = i\sum_A r_A^2 dz^A\wedge d\bar{z}^A \,. \tag{7.12}$$

In addition the perturbation

$$\delta J = i(dx^1\wedge dy^2 + dx^2\wedge dy^1) \sim dz^1\wedge d\bar{z}^2 + dz^2\wedge d\bar{z}^1 \tag{7.13}$$

satisfies $\delta J\wedge G = 0$. The remaining 5 Kähler moduli are lifted.

So far we have ensured that there is $\mathcal{N}=1$ supersymmetry. We will now argue that the solution above in fact preserves $\mathcal{N}=2$ supersymmetry.

Start by first taking the Kähler metric to be $g_{i\bar{j}} = \delta_{i\bar{j}}$. The coordinates x^i, y^i then define a flat coordinate system. Consider an SO(4)×U(1) subgroup of SO(6) where the SO(4) acts on the x^1, x^2, y^1, y^2, indices and the U(1) refers to rotations in the x^3, y^3, plane. It is easy to see that for the values (7.5), $G_{(3)}$ satisfies the relation, $\epsilon_{abcd} G_{(3)}^{cdl} = (G_{(3)})^{l}_{ab}$, and therefore transforms as a self dual representation of SO(4) (here we are following the notation of the previous section and the indices a, b take values x^1, x^2, y^1, y^2, while l, m range over x^3, y^3). Since we have already verified that $G_{(3)}$ satsifies the $\mathcal{N}=1$ conditions, it is ISD, and it follows that it must have charge 2 under the U(1). Putting all this together, in the example above we find that $G_{(3)}$ transforms as a $(0,3)_2$ representation under $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_L \times \mathrm{U}(1)$. As per our discussion above, it therefore meets the requirements for $\mathcal{N}=2$ supersymmetry.

Alternatively, working in the complex coordinates $z^i = x^i + iy^i, \bar{z}^i = x^i - iy^i$, let us define $A_{\bar{k}\bar{l}} = (G_{(3)})_{ij\bar{k}} \epsilon^{ij}_{\bar{l}}$. We see that $A_{\bar{1}\bar{1}}$ and $A_{\bar{2}\bar{2}}$ have nonzero values in the above example. Under the SU(3) symmetry, $(\bar{z}^1, \bar{z}^2, \bar{z}^3)$, transforms as a $\bar{3}$ representation. Consider an $\mathrm{SU}(2) \subset \mathrm{SU}(3)$ which acts on the $\bar{z}^1, \bar{z}^2$, coordinates and leaves $\bar{z}^3$ invariant. $A_{\bar{k}\bar{l}}$ or equivalently $G_{(3)}$ transforms as a triplet of this SU(2).

An additional check, also mentioned in section 4.1, is the following: in an $\mathcal{N}=2$ supersymmetric theory one should be able to define another inequivalent complex structure which keeps $G_{(3)}$ of kind $(2,1)$. In the example above it is easy to see that this corresponds to choosing holomorphic coordinates $(w^1, w^2, w^3) = (\bar{z}^1, \bar{z}^2, z^3)$.

Finally, some thought shows that under Kähler deformations of the form (7.12), (7.13), the conditions for $\mathcal{N}=2$ supersymmetry continue to hold.

8. Non-supersymmetric solutions

For generic (non-supersymmetric) solutions, we require only that the scalar potential vanish, or equivalently by (2.13), that $G_{(3)}$ be ISD. However, it is computationally simpler to consider the subclass of solutions in which $G_{(3)}$ is also primitive. In this case $G_{(3)}$ can only have pieces of type (2,1) and (0,3). The equations that one needs to solve are then

$$\begin{aligned} D_{\tau^{ij}} W &= \partial_{\tau^{ij}} W + (\partial_{\tau^{ij}} \mathcal{K}) W = 0\,, \\ D_\phi W &= \partial_\phi W + (\partial_\phi \mathcal{K}) W = 0\,, \end{aligned} \tag{8.1}$$

along with the primitivity condition,

$$J \wedge G_{(3)} = 0\,. \tag{8.2}$$

The first set of equations in (8.1) imposes the third set of equations appearing in (3.16), and forbids type (1,2) pieces of $G_{(3)}$. The second equation in (8.1) is the second equation in (3.16), i.e. forbids a (3,0) piece in $G_{(3)}$. Then, equation (8.2) kills the possibility of (2,1) IASD pieces in the three-form flux (T^6, unlike a generic Calabi-Yau, has a three-dimensional space of IASD non-primitive (2,1) forms). More generic non-supersymmetric solutions could be found by relaxing the requirement that the (1,2) ISD forms be absent from $G_{(3)}$, but we will not pursue them here.

Fluxes which obey the equations (8.1) and (8.2) will break supersymmetry iff $G_{(3)}$ contains a nontrivial component of type (0,3). This is easily interpreted in the low-energy supergravity: Since we are looking for solutions which are not necessarily supersymmetric, we no longer need to impose $D_{\rho^\alpha} W \propto W = 0$ for the Kähler moduli, ρ^α. Precisely when $G_{(3)}$ has a non-vanishing (0,3) piece, $W \neq 0$ and supersymmetry is broken, but still with vanishing potential (at leading order in α' and g_s). Examples of such vacua were discussed in [5, 28]. Such vacua will suffer a variety of instabilities in perturbation theory (as the "no-scale" structure of the potential will be violated by α' and g_s corrections), which is why we only discuss them briefly here.

The Kähler potential for the τ^{ij} is

$$\mathcal{K} = \mathcal{K}_{\text{dilaton}} + \mathcal{K}_{\text{cpx}} \,. \tag{8.3}$$

Here,

$$\mathcal{K}_{\text{dilaton}} = -\ln\bigl(-i(\phi - \bar{\phi})\bigr) \,, \tag{8.4}$$

and

$$\begin{aligned} \mathcal{K}_{\text{cpx}} &= -\ln\Bigl(-i \int_{T^6} \Omega \wedge \bar{\Omega}\Bigr) \\ &= -\ln \det\bigl(-i(\tau - \bar{\tau})\bigr) \\ &= -\ln\bigl(i\epsilon_{ijk}(\tau - \bar{\tau})^{i1}(\tau - \bar{\tau})^{j2}(\tau - \bar{\tau})^{k3}\bigr) \,. \end{aligned} \tag{8.5}$$

Since both τ^{ij} and $\bar{\tau}^{ij}$ enter into (8.1), it is in general difficult to solve the resulting non-holomorphic equations. However, in an ansatz with enough symmetry, the problem becomes tractable.

8.1 A non-supersymmetric example

Let us make a simple flux ansatz which is a subcase of the ansatz made in example 1 of section 4. We take $a^{ij} = a\delta^{ij}$, $d_{ij} = -a\delta_{ij}$, and b_0, c_0 to be nonzero, with all other fluxes vanishing. Then we find that the superpotential takes the form

$$\frac{1}{(2\pi)^2\alpha'} W = -c^0 \phi \det \tau - a^{ij} (\operatorname{cof} \tau)_{ij} + d_{ij} \phi \tau^{ij} - b_0 \,. \tag{8.6}$$

It is easy to compute the D3-charge carried by the fluxes with this ansatz,

$$N_{\text{flux}} = \frac{1}{(2\pi)^4(\alpha')^2} \int H_{(3)} \wedge F_{(3)} = b_0 c^0 - a^{ij} d_{ij} = b_0 c^0 + 3a^2 \,. \tag{8.7}$$

From the symmetry of the problem, one can show that $\tau^{ij} = \tau\delta^{ij}$. Let us further assume that

$$\tau = -\bar{\tau} \,, \qquad \phi = \tau \,. \tag{8.8}$$

Then,

$$\partial_\tau \mathcal{K} = -\frac{3}{2\tau} \,, \qquad \partial_\phi \mathcal{K} = -\frac{1}{2\tau} \,, \tag{8.9}$$

so that

$$\begin{aligned} D_\tau W &= \partial_\tau W + (\partial_\tau \mathcal{K}) W = -\frac{3}{2\tau}(c^0 \tau^4 - b_0) = 0 \,, \\ D_\phi W &= \partial_\phi W + (\partial_\phi \mathcal{K}) W = -\frac{1}{2\tau}(c^0 \tau^4 - b_0) = 0 \,. \end{aligned} \tag{8.10}$$

The equations are both satisfied if

$$\tau(=\phi) = i\left(\frac{b_0}{c^0}\right)^{1/4}, \tag{8.11}$$

therefore our assumption was consistent. Finally, since the flux ansatz is a special case of section 4 example 1, we can solve (8.2) by taking J to be in the same space that led to $G_{(3)}$ primitive in section 4.1. We can also check that the conditions for supersymmetry here are the same as those found earlier. The solution will be supersymmetric if $W = 0$. In the present example,

$$W = -6a\tau^2 - 2b_0 = 2b_0\left(\sqrt{\frac{9a^2}{b_0c^0}} - 1\right). \tag{8.12}$$

So, the solutions are non-supersymmetric as long as $9a^2 \neq b_0c^0$. In fact, it turns out there are no solutions which have even fluxes, $9a^2 = b_0c^0$ and $N_{\rm flux} \leq 32$ in any case.

9. Brane dynamics

In many of the examples of $\mathcal{N} = 1$ vacua with flux, one finds that the number of space-filling D3 branes needed to satisfy the tadpole cancellation requirement (2.6) is

$$N_{\rm D3} = 16 - \frac{1}{2}N_{\rm O3'} - \frac{1}{2(2\pi)^4(\alpha')^2}\int H_{(3)} \wedge F_{(3)} > 0 \tag{9.1}$$

($N_{D3} \geq 0$ is needed for supersymmetry). Therefore, in addition to the background 3-form flux, one must introduce space-filling D3 branes.

Following the work of Myers [29], it has been recognized that background p-form fields can have interesting effects on brane dynamics. It follows from [29] that the worldvolume potential (working at vanishing RR axion C_0) is given by

$$\mathcal{V}_{\rm open} \sim \frac{1}{g_s}H_{ijk}\,{\rm Tr}(X^iX^jX^k) - (*_6F_{(3)})_{ijk}\,{\rm Tr}(X^iX^jX^k) + \cdots \tag{9.2}$$

where $\cdots$ includes the usual $\mathcal{N} = 4$ field theory potential. When $G_{(3)}$ is ISD,

$$*_6F_{(3)} \;=\; \frac{1}{g_s}H_{(3)} \tag{9.3}$$

and the first two terms in (9.2) exactly cancel.

This is in keeping with the fact that the ISD fluxes mock up D3 brane charge and tension, and satisfy a "no force" condition with the D3 branes [5]. Therefore, at least at large radius (where supergravity intuition applies), the D3 point sources are free to live at arbitrary positions on the T^6. When $k \leq N_{D3}$ branes meet at a generic point, the low-energy physics is that of SU(k) $\mathcal{N} = 4$ SYM theory, while k branes meeting at an O3 plane will give rise to an SO($2N$) theory, as usual. It would be interesting to determine the leading nontrivial effects of the fluxes on the D-branes, and to find more elaborate types of models where phenomena reminiscent of those observed in [30] can occur. Inclusion of anti-branes in the flux background might also lead to interesting phenomena, as in [19].

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It follows from this discussion that inclusion of N_{D3} branes in one of our models adds $3N_{D3}$ complex moduli to the low energy theory. From this perspective, the models with $N_{\rm flux} \simeq 32$ and $N_{D3} \simeq 0$ are the most satisfying.

10. Discussion

IIB compactifications on Calabi-Yau spaces with both RR and NS 3-form fluxes turned on provide a rich class of vacua which are amenable to detailed study. It should be clear that the techniques used here to compute W and study vacua of the T^6/Z_2 orientifold would generalize to many other examples. The main novelty of these examples is that they provide a setting where the stabilization of Calabi-Yau moduli becomes a concrete and tractable problem. These models are also of interest because they give rise to warped compactifications of string theory, and in some cases the low-energy physics has a holographic interpretation via variants of the AdS/CFT duality [9, 5].

Several natural questions about the T^6/Z_2 models studied here would be suitable for further study. A complete classification of supersymmetric vacua may be possible (although, especially in cases where the additional complications of discrete RR and NS flux arise [15], it could be very difficult to achieve). It is also interesting to ask whether there are any cases where, with a fixed topological class for the fluxes, one finds multiple vacua. Finally, various dual descriptions of these models should exist, and fleshing out these dualities (and in particular, understanding any analogues of mirror symmetry for vacua with nonzero H-flux) seems worthwhile.

11. Note added for JHEP version

Since this paper is being published substantially after its initial date of release, it was deemed appropriate to add a brief discussion of progress that has occurred in related work in the last 18 months. There have actually been advances on several related fronts.

The precise models discussed in this paper and in the related work of [15] have been further investigated in several directions. Their low energy description as gauged $\mathcal{N}=4$ supergravities of a novel type has been worked out in [31], and some cosmological solutions based on these models were given in [32]. The connection between the flux sectors with different amounts of 4d supersymmetry was investigated in [33], where it was found that models with different amounts of SUSY can nevertheless be connected by going slightly off shell. Finally, it has been shown that these models admit dual descriptions involving compactifications of type-II strings on non-Kähler spaces with background flux [34].

In addition, there has been much work on extending the kinds of ideas discussed in this paper to backgrounds of greater physical interest. On one such front, several groups have found that it is possible to write down type-IIB models which have flux stabilization of the moduli and admit D-branes with chiral gauge theories on their worldvolume [35]. This is of obvious potential interest for particle physics model building. On another front, it has been demonstrated that flux compactifications in the presence of rather generic non-perturbative effects can give rise to vacua with no moduli at all. In the simplest cases these

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are AdS vacua, while more elaborate constructions yield de Sitter vacua [36]. Approaches to give a microscopic computation of the de Sitter entropy are now being developed in this framework [37], as are attempts to find stringy models of inflation [38].

In an effort to get a better global view of the resulting space of string vacua, a program has been initiated to "count" vacua by approximating the flux superpotentials and making suitable assumptions of genericity. This has yielded estimates for the precise number of flux vacua in very simple examples [39].

Clearly, further developments along all of these lines may be expected. A review of these efforts has recently appeared in [40].

Acknowledgments

We would like to thank S. Das, K. Dasgupta, S. Gukov, R. Kallosh, N. Kaloper, J. Polchinski, S. Sethi, E. Silverstein, P. Tripathi and H. Verlinde for useful discussions about related subjects. S.T. is particularly grateful to A. Dabholkar for several helpful comments and insights and to T. R Ramadas for bringing [41] to his attention and for related discussions. S.K. enjoyed the hospitality of the Weizmann Institute of Science and the Amsterdam Summer Workshop on String Theory during the course of this work. This research was supported in part by the Department of Energy under contract DE-AC03-76SF00515. The work of S.K. is also supported by a Packard Fellowship, a Sloan Fellowship, and National Science Foundation grant PHY-0097915. S.T. acknowledges support from the IMSC String workshop, 2001, where some of this work was done.

A. Flux quantization

We follow the conventions of [5] and [16]. A Dp-brane couples to the $(p+2)$-form RR field strength via the action

$$-\frac{1}{2\kappa_{10}^2}\frac{1}{2(p+2)!}\int_{\mathcal{M}_6} d^{10}x\sqrt{-g}\; {F_{(p+2)}}^2 + \mu_p \int C_{p+1}\,. \tag{A.1}$$

The usual quantization condition that follows from this action is

$$\int_\gamma F_{p+2} = \left(2{\kappa_{10}}^2\mu_{6-p}\right)n_\gamma\,, \qquad n_\gamma \in \mathbb{Z}\,, \qquad \mu_p = \frac{1}{(2\pi)^p}\alpha'^{-\frac{p+1}{2}}\,, \tag{A.2}$$

for an arbitrary 3-cycle $\gamma \in H_3(\mathcal{M}_6, \mathbb{Z})$. Here μ_p is the electric charge of a Dp-brane and μ_{6-p} is the charge of the dual magnetic D$(6-p)$-brane. The product of these two charges is related to the factor $1/2{\kappa_{10}}^2 = (2\pi)^7\alpha'^4$ that multiplies the action, via the Dirac quantization condition

$$\mu_p\mu_{6-p} = \frac{2\pi}{2{\kappa_{10}}^2}\,. \tag{A.3}$$

From (A.2) and (A.3),

$$\mu_p \int F_{p+2} = 2\pi n\,, \qquad n \in \mathbb{Z}\,, \tag{A.4}$$

which, in the case $p=1$, becomes

$$\frac{1}{2\pi\alpha'}\int F_3 = 2\pi n\,, \qquad n\in\mathbb{Z}\,. \tag{A.5}$$

Similarly, we know that the electric NS charge of a fundamental string is $\mu_{\rm F1} = 1/2\pi\alpha'$. So, using $\mu_{\rm F1}\mu_{\rm NS5} = 2\pi/2\kappa_{10}{}^2$ together with the analog of the first equation in (A.2),

$$\frac{1}{2\pi\alpha'}\int H_3 = 2\pi n\,, \qquad n\in\mathbb{Z}\,. \tag{A.6}$$

This equation can also be obtained from (A.5) by S-duality.

For compactification on T^6/Z_2, it can be shown that the quantization condition is exactly (A.2), with $\mathcal{M}_6 = T^6$ [15].[16] The 3-cycles on T^6/Z_2 include both the 3-cycles on T^6 and also new cycles, such as

$$\gamma_0: \qquad 0\le x^1, x^2 \le 1\,, \quad 0\le x^3\le \frac{1}{2}\,, \quad y^i = 0\,, \tag{A.7}$$

which are "half-cycles" on T^6. Naively, this would seem to lead to a problem with the quantization condition (A.2). Define γ_1 by

$$\gamma_1: \qquad 0\le x^1, x^2, x^3 \le 1\,, \quad y^i = 0\,. \tag{A.8}$$

Then, one has $n_{\gamma_0} = \frac{1}{2}n_{\gamma_1}$, so that $n_{\gamma_0}\notin\mathbb{Z}$ when n_{γ_1} is odd. However, as discussed in [15], the quantization condition is still satisfied in this case, if a half unit of discrete RR flux is turned on at an odd number of the O3-planes that intersect $\gamma_{0,1}$. Similarly, when m_{γ_1} is odd, a half unit of NS flux must be turned on at an odd number of the O3-plane that intersects $\gamma_{0,1}$. When n_{γ_1} (m_{γ_1}) is even, it is also permissible to turn on RR (NS) flux at some of the O3-planes that intersect $\gamma_{0,1}$, but we require that the total number of such O3-planes be even. Because the construction of vacua with these exotic O3 planes is somewhat involved except in the simplest examples, we have focused in this paper on cases where all of the fluxes in the covering space are even integers, and the naive problem does not arise.

B. Derivation of equation (4.76)

Write $\tau^{ij} = T^{ij} + A^{ij}/A^0 = T^{ij} + \widetilde{A}^{ij}$, where a tilde denotes division by A^0. Here, $A^{ij} = a^{ij} - \phi c^{ij}$, and B_{ij}, A^0 and B_0 are defined similarly. Then, equation (3.22c) becomes

$$\widetilde{W} = \det\tau - \widetilde{A}^{ij}(\operatorname{cof}\tau)_{ij} - \widetilde{B}_{ij}\tau^{ij} - \widetilde{B}_0\,, \tag{B.1}$$

which, after some algebra can be shown to have the T^{ij} expansion

$$\widetilde{W} = \det T - \big((\operatorname{cof}\widetilde{A})_{ij} + \widetilde{B}_{ij}\big)T^{ij} - \big(\widetilde{A}^{ij}\widetilde{B}_{ij} + \widetilde{B}_0 + 2\det\widetilde{A}\big)\,.$$

The analog of equation (3.22c) has already been obtained in equation (4.75),

$$(\operatorname{cof} T)_{ij} = (\operatorname{cof}\widetilde{A})_{ij} + \widetilde{B}_{ij}\,. \tag{B.2}$$

[16] We are indebted to A. Frey and J. Polchinski for providing us with a preliminary draft of their preprint [15]. The remainder of this section summarizes an analogous section in their preprint.

By virtue of this equation, the previous result becomes

$$\widetilde{W} = -2\det T + (\widetilde{A}^{ij}\widetilde{B}_{ij} + \widetilde{B}_0 + 2\det\widetilde{A})\,.$$

When $W = 0$,

$$\det T = -\frac{1}{2}(\widetilde{A}^{ij}\widetilde{B}_{ij} + \widetilde{B}_0 + 2\det\widetilde{A})\,. \tag{B.3}$$

Since we have independent expressions (B.2) and (B.3) for $\mathrm{cof}\,T$ and $\det T$, respectively, the equality

$$\det\mathrm{cof}\,T = (\det T)^2 \tag{B.4}$$

gives a quartic equation for ϕ. Explicitly, we have

$$\begin{aligned}\det\mathrm{cof}\,T &= \det\mathrm{cof}\,\widetilde{A} + (\mathrm{cof}\,\mathrm{cof}\,\widetilde{A})^{ij}\widetilde{B}_{ij} + (\mathrm{cof}\,\widetilde{A})_{ij}(\mathrm{cof}\,\widetilde{B})^{ij} + \det\widetilde{B}\\ &= (\det\widetilde{A})^2 + (\det\widetilde{A})(\widetilde{A}^{ij}\widetilde{B}_{ij}) + (\mathrm{cof}\,\widetilde{A})_{ij}(\mathrm{cof}\,\widetilde{B})^{ij} + \det\widetilde{B}\,,\end{aligned} \tag{B.5}$$

and

$$(\det T)^2 = (\det\widetilde{A})^2 + (\det\widetilde{A})(\widetilde{A}^{ij}\widetilde{B}_{ij} + \widetilde{B}_0) + \frac{1}{4}(A^0B_0 + A^{ij}B_{ij})^2\,, \tag{B.6}$$

so, equating the two and multiplying by $(A^0)^4$,

$$(\det A)B_0 - (\det B)A^0 + (\mathrm{cof}\,A)_{ij}(\mathrm{cof}\,B)^{ij} + \frac{1}{4}(A^0B_0 + A^{ij}B_{ij})^2 = 0\,, \tag{B.7}$$

which is equation (4.76).

C. Derivation of equation (6.9)

To establish that (6.9), is correct, notice first that the transformed superpotential for the new fluxes is given by

$$W'[\tau] = \int G'_{(3)} \wedge \Omega[\tau]\,, \tag{C.1}$$

where we have explicitly indicated that the dependence on the complex structure moduli arises from Ω on the right hand side. Using, (6.7), (C.1), can also be expressed as

$$W'[\tau] = (G_{(3)})_{rst}\Omega[\tau]_{def}M^r_aM^s_bM^t_c\epsilon^{abcdef}\,. \tag{C.2}$$

Now,

$$\Omega[\tau] = dz^1 \wedge dz^2 \wedge dz^3\,, \tag{C.3}$$

where,

$$\begin{pmatrix} dz^1\\ dz^2\\ dz^3\end{pmatrix} = (1,\tau)\cdot\begin{pmatrix} dx^i\\ dy^i\end{pmatrix}. \tag{C.4}$$

Under a change of complex structure, $\tau \to \tau'$ (where τ' is given by (6.8))

$$\begin{pmatrix} dz^1\\ dz^2\\ dz^3\end{pmatrix} \to \begin{pmatrix} (dz^1)'\\ (dz^2)'\\ (dz^3)'\end{pmatrix} = N(1,\tau)M\cdot\begin{pmatrix} dx^i\\ dy^i\end{pmatrix}. \tag{C.5}$$

As a result one finds that[17]

$$\Omega[\tau']_{(def)} = \det(N)\Omega[\tau]_{uvw} M^u_d M^e_b M^w_f \,. \tag{C.6}$$

Substituting in (C.2), then leads to

$$W'[\tau'] = \det(N)\det(M) W[\tau]\,. \tag{C.7}$$

D. The spinor conditions for $\mathcal{N} = 2$ supersymmetry

Throughout this appendix the components for all tensors will be evaluated in a vielbein frame. We will also use the notation introduced in section 7. The SO(6) group of rotations in the 6 compactified directions has an SO(4) $\times$ U(1) subgroup. In our notation, indices a, b which take four values refer to directions which transform under the SO(4) and indices l, m which take two values refer to directions which are acted on by the U(1) subgroup. The metric in the vielbein frame has components $g_{ab} = \delta_{ab}$, $g_{lm} = \delta_{lm}$, $g_{al} = 0$. Also the γ matrices satisfy the relation

$$\{\gamma^l, \gamma^a\} = 0\,. \tag{D.1}$$

Using the fact that an $\mathrm{SU}(2)_R$ symmetry group must be left unbroken we argued in section 7 that the flux must have the index structure, $(G_{(3)})_{(abl)}$, and further that $G_{(3)}$ must transform as $(0,3)_{\pm 2}$ under the $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_L \times \mathrm{U}(1) \subset \mathrm{SO}(6)$ group. Here we will show that the spinor conditions imply that the $(0,3)_{-2}$ terms must be absent and $G_{(3)}$ must only transform as a $(0,3)_2$ representation under this group.

The spinor conditions are given in [14] and (3.8),

$$G_{(3)}\epsilon = G_{(3)}\epsilon^* = G_{(3)}\gamma^l\epsilon^* = G_{(3)}\gamma^a\epsilon^* = 0\,. \tag{D.2}$$

In our choice of conventions, the spinor 4 representation of SO(6) transforms as $(2,1)_1 + (1,2)_{-1}$ under $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_L \times \mathrm{U}(1)$. In the $\mathcal{N} = 2$ supersymmetry case, ϵ is a doublet of $\mathrm{SU}(2)_R$ and therefore transforms as a $(2,1)_1$ representation of $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_L \times \mathrm{U}(1)$. We are now ready to ask what conditions (D.2) imposes on the flux $G_{(3)}$.

We noted above that the flux has index structure G_{abl}. Using (D.1), the first condition in (D.2) can be explicitly written as

$$(G_{(3)})_{lab}\gamma^l[\gamma^a, \gamma^b]\epsilon = 0\,. \tag{D.3}$$

If $G_{(3)}$ transforms as $(0,3)_{\pm 2}$ under $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_L \times \mathrm{U}(1)$ it is easy to see that $(G_{(3)})_{lab}[\gamma^a, \gamma^b]$ is a generator of $\mathrm{SU}(2)_L$ and therefore must annihilate ϵ, which is a singlet of $\mathrm{SU}(2)_L$. So (D.3) is met.

Similarly, since ϵ^* is also a singlet under $\mathrm{SU}(2)_L$, it is also true that $G_{(3)}\epsilon^* = 0$. The third condition in (D.2), can be written as

$$\frac{1}{2}(G_{(3)})_{mab}\gamma^m\gamma^l[\gamma^a, \gamma^b]\epsilon^* = 0\,. \tag{D.4}$$

[17] This follows, for example, by noting from (C.5) that, up to an overall normalization of $\det(N)$, $\Omega[\tau']$ in the basis $\binom{(dx^i)'}{(dy^i)'} = M\binom{dx^i}{dy^i}$ has the same components as $\Omega[\tau]$ in the basis $\binom{dx^i}{dy^i}$.

Once again the same argument leading to the first two conditions being met ensures that (D.4) is also satisfied.

Finally we come to the last condition in (D.2). This can be expressed as

$$(G_{(3)})_{bcl}\gamma^b\gamma^c\gamma^a\gamma^l\epsilon^* = 0\,. \tag{D.5}$$

One can show that (D.5) is not met if $G_{(3)}$ has a $(0,3)_{-2}$ component. If this component is absent though, and $G_{(3)}$ is entirely of the $(0,3)_2$ kind, one can show that

$$(G_{(3)})_{bcl}\gamma^l\epsilon^* = 0\,. \tag{D.6}$$

Condition (D.5) then follows.

To show that (D.6) is satisfied when $G_{(3)}$ transforms as a $(0,3)_2$ state we first note, as was pointed out above, that ϵ has charge $+1$ with respect to the U(1). So ϵ^* has charge -1. As a result, if $G_{(3)}$ is of $(0,3)_2$ kind, $(G_{(3)})_{abl}\gamma^l\epsilon^*$, has charge -3 under the U(1). Also note that the state $(G_{(3)})_{abl}\gamma^l\epsilon^*$ transforms as a 4 spinor under the SO(6) symmetry. But the 4 representation does not have any state with -3 charge under the U(1) symmetry. Thus the left hand side of (D.6) must vanish.

In summary, the spinor conditions show that $G_{(3)}$ must transform under $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_L \times \mathrm{U}(1)$ as a $(0,3)_2$ representation, in order to preserve $\mathcal{N}=2$ supersymmetry.

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